



FORGED MATHEMATICS SERIES · VOLUME I

ALGEBRA

EXPRESSIONS · EQUATIONS · FUNCTIONS

MATHEMATICS

FORGED OFFICIAL TEXTBOOK

A twelve-chapter course from arithmetic to functions: expressions and exponents; linear equations, inequalities, and graphs; systems; polynomials and factoring; quadratics; rational and radical expressions; and exponential and logarithmic functions.

First edition · Theorems, proofs, worked examples, and verified exercises



FORGED MATHEMATICS SERIES • VOLUME I

ALGEBRA

EXPRESSIONS • EQUATIONS • FUNCTIONS

Omnistrata, LLC

First Edition

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Learn skills that matter.

Algebra

ForgEd Mathematics Series, Volume I

First Edition

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The Forged Mathematics Series

Volume I Algebra

Volume II Precalculus & Trigonometry

Volume III Calculus I & II

Volume IV Multivariable Calculus & Quantum Mechanics

Typeset in TeX Gyre Pagella and Cinzel.

PREFACE

This book is a complete course in algebra, from the arithmetic of signed numbers and fractions to exponential and logarithmic functions. It assumes only arithmetic. It is the first volume of the Forged Mathematics Series, and it is written so that a reader who finishes it is ready for Volume II, *Precalculus & Trigonometry*, and through it for calculus.

Algebra is often taught as a list of procedures to memorize. This book takes a different view: every procedure is a consequence of a small number of properties of the real numbers, and a student who sees where a rule comes from can rebuild it when memory fails and can tell when it does not apply. So the reasons are given alongside the rules, in plain language, and the most common errors are named and explained where they happen.

HOW THE BOOK IS ORGANIZED

Part I (Chapters 1–3) lays the foundations: the real numbers and their arithmetic, algebraic expressions and the laws of exponents, and linear equations and inequalities in one variable. **Part II** (Chapters 4–6) studies linear relationships: the coordinate plane and lines, the idea of a function, and systems of linear equations and inequalities. **Part III** (Chapters 7–9) covers polynomials, factoring, and quadratic equations and functions. **Part IV** (Chapters 10–12) treats rational expressions, radicals and rational exponents, and gives a first look at exponential and logarithmic functions.

HOW TO READ THIS BOOK

Each chapter opens with its learning objectives. Boxed material comes in four kinds.

Definition 0.1 (Definitions, theorems, properties). Shaded boxes contain the statements you should be able to reproduce from memory. Theorems

carry a gold rule.

Example 0.2 (Examples). Worked examples. Cover the solution and attempt the problem before reading it.

COMMON ERROR. Black-ruled boxes describe the specific mistakes students make most often, and how to detect them.

LOOKING AHEAD. Dark boxes connect the current material to where it is used later in this book or in later volumes.

Theorems, definitions, and examples share one numbering sequence within each chapter.

PRACTICE

Each chapter ends with three kinds of practice. *Worked Problems* are fully solved problems that combine the chapter's ideas, a step harder than the examples. *Skill Drills* are short problems for speed and accuracy, with answers immediately after the list. *Exercises* are grouped as *Computation*, *Concepts and Proof*, and *Challenge*; answers to odd-numbered exercises with a short final answer are at the back of the book. Every numerical answer in the examples, worked problems, drills, and answer section was checked by computer algebra before publication. If your result disagrees, find the step where it diverges.

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PART I

FOUNDATIONS

The real numbers and their arithmetic, algebraic expressions and exponents, and linear equations and inequalities in one variable.

CHAPTER 1

NUMBERS AND ARITHMETIC

LEARNING OBJECTIVES

- Classify real numbers as natural, whole, integer, rational, or irrational, and place them on the number line.
- Compute with signed numbers, fractions, and decimals without a calculator.
- Apply the order of operations and the properties of the real numbers.
- Use absolute value to measure distance, and solve problems with percents, ratios, and proportions.

Algebra is arithmetic with letters standing in for numbers. Every rule of algebra in this book is a rule of arithmetic that holds for *all* numbers, so it pays to be precise about what the numbers are and how they combine. This chapter reviews that arithmetic carefully. Much of it will be familiar; read it anyway for the reasons behind the rules, which the rest of the book uses constantly.

1.1 THE REAL NUMBERS

Definition 1.1 (The number systems). • The *natural numbers* are $\mathbb{N} = \{1, 2, 3, \dots\}$; the *whole numbers* are $\{0, 1, 2, 3, \dots\}$.

- The *integers* are $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- The *rational numbers* $\mathbb{Q}$ are the numbers that can be written as a quotient $\frac{p}{q}$ of integers with $q \neq 0$.
- A real number that is not rational is *irrational*. Together, the rational and irrational numbers form the *real numbers* $\mathbb{R}$.

Every integer is rational ($-5 = \frac{-5}{1}$), and every natural number is an integer, so the sets are nested as in Fig. 1.1. A rational number has a decimal expansion

sion that either terminates ($\frac{3}{8} = 0.375$) or repeats forever in a fixed block ($\frac{4}{11} = 0.363636\ldots = 0.\overline{36}$). An irrational number has a decimal expansion that does neither. The best-known irrational numbers are $\sqrt{2}$, $\sqrt{3}$ and π .

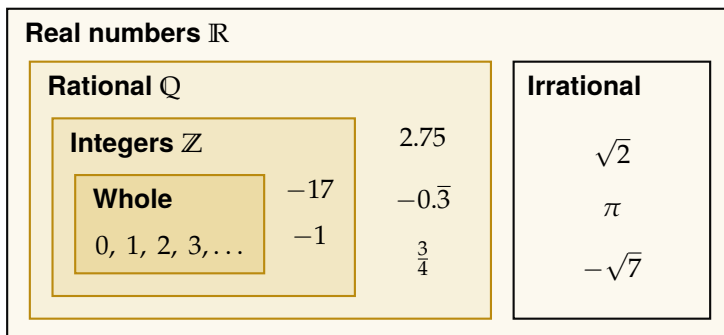


Figure 1.1: The real numbers. The natural numbers are the whole numbers other than 0. Every real number is either rational or irrational, never both.

Example 1.2 (Classifying numbers). Classify each number: -7 , 0 , $\frac{12}{4}$, $\sqrt{5}$, $0.\overline{1}$, -2.6 , $\sqrt{9}$, $\frac{\pi}{2}$.

Solution. $\frac{12}{4} = 3$ and $\sqrt{9} = 3$ are natural numbers (so also whole, integer and rational). 0 is whole. -7 is an integer. $0.\overline{1} = \frac{1}{9}$ and $-2.6 = -\frac{13}{5}$ are rational but not integers. $\sqrt{5}$ and $\frac{\pi}{2}$ are irrational. Simplify a number before classifying it: $\sqrt{9}$ looks like a radical but is the integer 3.

How do we know that irrational numbers exist at all? The ancient Greeks proved it, and the proof is a model of mathematical reasoning.

Theorem 1.3 ($\sqrt{2}$ is irrational). *There are no integers p and q with $\left(\frac{p}{q}\right)^2 = 2$.*

Proof. Suppose there were. Cancel common factors so that $\frac{p}{q}$ is in lowest terms. Then $p^2 = 2q^2$, so p^2 is even. The square of an odd number is odd, so p itself is even: $p = 2k$. Substituting, $4k^2 = 2q^2$, so $q^2 = 2k^2$ and q is even too. But then p and q share the factor 2, contradicting lowest terms. So no such fraction exists. ■

1.1.1 THE NUMBER LINE AND ABSOLUTE VALUE

The real numbers correspond to the points of a line: choose a point for 0 and a unit length, put positive numbers to the right and negative numbers to the left.

We write $a < b$ (a is less than b) when a lies to the left of b ; $a \leq b$ means $a < b$ or $a = b$.

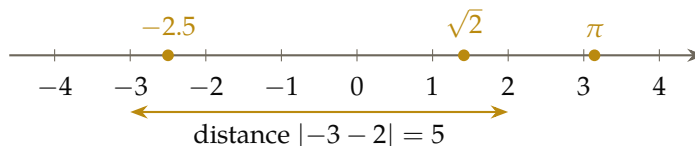


Figure 1.2: Real numbers as points on a line. The distance between a and b is $|a - b|$.

Definition 1.4 (Absolute value). The *absolute value* of a is

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}$$

It is the distance from a to 0 on the number line. The distance between a and b is $|a - b|$.

So $|6| = 6$, $|-6| = -(-6) = 6$, and the distance between -3 and 2 is $|-3 - 2| = 5$.

COMMON ERROR ($-a$ is not always negative). The symbol $-a$ means “the opposite of a .” If $a = -4$, then $-a = 4$ is positive. This is why the definition of $|a|$ uses $-a$ when $a < 0$: it turns a negative number into a positive one.

1.2 SIGNED NUMBERS

The rules for adding and multiplying signed numbers come from thinking of $-a$ as the number that cancels a : $a + (-a) = 0$.

Theorem 1.5 (Arithmetic of signed numbers). (1) To add numbers with the same sign, add their absolute values and keep the sign. To add numbers with opposite signs, subtract the smaller absolute value from the larger and take the sign of the one with larger absolute value.

(2) Subtraction is adding the opposite: $a - b = a + (-b)$.

(3) A product or quotient of two numbers with the same sign is positive; with opposite signs it is negative.

(4) $0 \cdot a = 0$, and $\frac{0}{a} = 0$ for $a \neq 0$. Division by 0 is undefined.

Why is a negative times a negative positive? Start from $0 = (-3) \cdot 0 = (-3)(4 + (-4)) = (-3)(4) + (-3)(-4) = -12 + (-3)(-4)$. The only number that adds to -12 to give 0 is 12 , so $(-3)(-4) = 12$. The rule is forced by the distributive property, not chosen by convention.

Why can't we divide by zero? A quotient $\frac{a}{b} = q$ means $a = bq$. If $b = 0$ and $a \neq 0$, no q works, since $0 \cdot q = 0$. If $a = b = 0$, every q works, so no single answer can be chosen. Either way, $\frac{a}{0}$ has no value.

Example 1.6 (Signed arithmetic). (a) $-8 + 5 = -3$. (b) $-3 - (-9) = -3 + 9 = 6$. (c) $(-4)(-6)(-1) = 24(-1) = -24$. (d) $\frac{-45}{-9} = 5$. (e) $(-2)^4 = 16$, because four negative factors give a positive product.

COMMON ERROR (-2^4 versus $(-2)^4$). An exponent applies only to what is immediately to its left. $-2^4 = -(2^4) = -16$, while $(-2)^4 = 16$. The parentheses decide which is meant.

1.3 FRACTIONS AND DECIMALS

Two fractions are *equivalent* if they represent the same number: $\frac{a}{b} = \frac{ac}{bc}$ for any $c \neq 0$. A fraction is in *lowest terms* when its numerator and denominator have no common factor other than 1. To reduce a fraction, divide both parts by their *greatest common divisor* (GCD), which can be read from prime factorizations.

Example 1.7 (Lowest terms). Reduce $\frac{84}{126}$.

Solution. $84 = 2^2 \cdot 3 \cdot 7$ and $126 = 2 \cdot 3^2 \cdot 7$. The common factors are $2 \cdot 3 \cdot 7 = 42$, so $\frac{84}{126} = \frac{2}{3}$.

Theorem 1.8 (Operations on fractions). For integers with nonzero denominators:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \quad (c \neq 0).$$

In practice, add fractions over the *least common denominator* (LCD), the least common multiple of the denominators. This keeps the numbers small.

Example 1.9 (Fraction arithmetic). (a) $\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}$. (b) $\frac{2}{3} \div \frac{4}{9} = \frac{2}{3} \cdot \frac{9}{4} = \frac{18}{12} = \frac{3}{2}$. (c) $2\frac{1}{3} \cdot 1\frac{1}{2} = \frac{7}{3} \cdot \frac{3}{2} = \frac{7}{2} = 3\frac{1}{2}$. Convert mixed numbers to fractions before multiplying.

COMMON ERROR (Adding tops and bottoms). $\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$. A quick check exposes the error: $\frac{2}{5}$ is less than $\frac{1}{2}$, but adding a positive number must make $\frac{1}{2}$ larger. The correct sum is $\frac{5}{6}$.

1.3.1 DECIMALS

To write a fraction as a decimal, divide: $\frac{3}{8} = 0.375$ and $\frac{5}{12} = 0.41666\ldots = 0.41\overline{6}$. To write a repeating decimal as a fraction, multiply to shift the repeating block and subtract.

Example 1.10 (A repeating decimal as a fraction). Write $0.\overline{36}$ as a fraction.

Solution. Let $x = 0.363636\ldots$. Then $100x = 36.363636\ldots$, and subtracting cancels the infinite tails: $99x = 36$, so $x = \frac{36}{99} = \frac{4}{11}$.

1.4 ORDER OF OPERATIONS AND PROPERTIES

An expression such as $18 \div 3 \cdot 2 - 4^2$ needs an agreed reading. The standard one is:

Definition 1.11 (Order of operations). (1) Simplify inside grouping symbols (parentheses, brackets, fraction bars, absolute value bars), innermost first.

- (2) Evaluate powers.
- (3) Multiply and divide from left to right.
- (4) Add and subtract from left to right.

Example 1.12 (Applying the order). (a) $18 \div 3 \cdot 2 - 4^2 + (5 - 8)^2 = 6 \cdot 2 - 16 + 9 = 5$.

(b) $\frac{-3 + 2(4 - 7)^2}{5 - 2 \cdot 4} = \frac{-3 + 2 \cdot 9}{5 - 8} = \frac{15}{-3} = -5$. The fraction bar groups the numerator and the denominator.

COMMON ERROR (Left to right means left to right). $12 \div 2 \cdot 3 = 6 \cdot 3 = 18$, not $12 \div 6 = 2$. Multiplication does not come before division; they share a level and are done left to right.

Theorem 1.13 (Properties of the real numbers). For all real numbers a, b, c :

Property	Addition	Multiplication
Commutative	$a + b = b + a$	$ab = ba$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	$a + 0 = a$	$a \cdot 1 = a$
Inverse	$a + (-a) = 0$	$a \cdot \frac{1}{a} = 1 \quad (a \neq 0)$
Distributive	$a(b + c) = ab + ac$	

These properties are the rules of algebra. Every manipulation in later chapters, from combining like terms to factoring, is a chain of them. The distributive property is the one used most: it is the only rule that connects addition with multiplication.

Example 1.14 (Mental arithmetic with the distributive property). $7 \cdot 98 = 7(100 - 2) = 700 - 14 = 686$, and $23 \cdot 17 + 23 \cdot 83 = 23(17 + 83) = 2300$.

1.5 PERCENTS, RATIOS, AND PROPORTIONS

Percent means “per hundred”: $p\% = \frac{p}{100}$. So $35\% = 0.35$ and $0.065 = 6.5\%$. Most percent problems are one of three questions about the equation $\text{part} = \text{rate} \times \text{whole}$.

Definition 1.15 (Percent change). When a quantity changes from an old value to a new value, the *percent change* is $\frac{\text{new} - \text{old}}{\text{old}} \times 100\%$. Increasing a quantity by $r\%$ multiplies it by $1 + \frac{r}{100}$; decreasing it by $r\%$ multiplies it by $1 - \frac{r}{100}$.

Example 1.16 (Discount and tax). An \$80 jacket is discounted 25%, and 7% sales tax is added to the sale price. What is the final cost?

Solution. Multiplying by the factors: $80(0.75)(1.07) = 60(1.07) = 64.20$.

The final cost is \$64.20.

COMMON ERROR (Percent changes do not add). A price that rises 20% and then falls 20% does not return to its starting value. The factors multiply: $(1.20)(0.80) = 0.96$, a net *decrease* of 4%. The second change is a percent of a larger base.

A *ratio* compares two quantities by division, and a *proportion* states that two ratios are equal. For $b, d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc,$$

which is called *cross-multiplication*; multiply both sides by bd to see why.

Example 1.17 (Scaling a recipe). A recipe uses 3 cups of flour for 24 cookies. How much flour is needed for 40 cookies?

Solution. $\frac{3}{24} = \frac{x}{40}$, so $24x = 120$ and $x = 5$ cups.

LOOKING AHEAD. The distinction between rational and irrational numbers returns in Volume III: the real numbers have no gaps, and this *completeness* is what makes limits and calculus work. Absolute value as distance is the basis of the inequalities $|x - a| < \delta$ that define limits. And the multiply-the-factors view of percent change is exactly exponential growth, studied in Chapter 12.

1.6 WORKED PROBLEMS

These problems combine the skills of the chapter. Work each one before reading the solution, and check every answer with a quick estimate.

PROBLEM 1.1 A fraction expression

Simplify $(\frac{2}{3} - \frac{3}{4}) \div \frac{5}{6} + \frac{1}{2}$.

Solution. Grouping first: $\frac{2}{3} - \frac{3}{4} = \frac{8}{12} - \frac{9}{12} = -\frac{1}{12}$. Then divide: $-\frac{1}{12} \cdot \frac{6}{5} = -\frac{6}{60} = -\frac{1}{10}$. Finally $-\frac{1}{10} + \frac{1}{2} = -\frac{1}{10} + \frac{5}{10} = \frac{4}{10} = \frac{2}{5}$.

PROBLEM 1.2 A decimal with a delayed repeat

Write $2.\overline{417} = 2.41717\dots$ as a fraction.

Solution. Let $x = 2.41717\dots$. The repeating block has two digits and begins one place after the decimal point. Shift the block start to the decimal point and then one full block further: $10x = 24.1717\dots$ and $1000x = 2417.1717\dots$. Subtracting, $990x = 2393$, so $x = \frac{2393}{990}$. The fraction is in lowest terms, since $990 = 2 \cdot 3^2 \cdot 5 \cdot 11$ and none of these primes divides 2393.

PROBLEM 1.3 Three percent changes

A stock rises 25%, then falls 20%, then rises 10%. What is its overall percent change?

Solution. Multiply the factors: $(1.25)(0.80)(1.10) = 1.10$. The overall change is an increase of 10%. Adding the percents ($25 - 20 + 10 = 15$) gives the wrong answer, because each change applies to a different base.

PROBLEM 1.4 Working backward from a total

After 8% sales tax, a bill comes to \$91.80. What was the price before tax?

Solution. If the price is P , then $1.08P = 91.80$, so $P = \frac{91.80}{1.08} = 85$. The price was \$85. Taking 8% off the total ($91.80 \times 0.92 = 84.46$) is wrong: the tax was 8% of the price, not of the total.

PROBLEM 1.5 Converting units

Convert 60 miles per hour to feet per second.

Solution. Multiply by conversion factors equal to 1, arranged so the unwanted units cancel:

$$\frac{60 \text{ mi}}{1 \text{ h}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = \frac{60 \cdot 5280 \text{ ft}}{3600 \text{ s}} = 88 \frac{\text{ft}}{\text{s}}.$$

PROBLEM 1.6 Ordering close numbers

Order $\frac{5}{8}$, 0.62 , $\frac{3}{5}$, and $\sqrt{0.4}$ from least to greatest.

Solution. Convert each to a decimal: $\frac{5}{8} = 0.625$, $\frac{3}{5} = 0.6$, and $\sqrt{0.4} \approx 0.632$ (since $0.632^2 \approx 0.399$). The order is $\frac{3}{5} < 0.62 < \frac{5}{8} < \sqrt{0.4}$.

PROBLEM 1.7 When do schedules line up?

One bus leaves a station every 12 minutes and another every 18 minutes. Both leave at 8:00. When do they next leave together?

Solution. They leave together at times that are multiples of both 12 and 18, so the next time is their least common multiple. $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$, so the LCM is $2^2 \cdot 3^2 = 36$. They next leave together at 8:36.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D1.1 $-9 + 4$

D1.2 $-7 - 8$

D1.3 $6 - (-5)$

D1.4 $(-6)(-7)$

D1.5 $-63 \div 9$

D1.6 $(-1)^{10}$

D1.7 -5^2

D1.8 $|-13| + |5|$

D1.9 $|4 - 11|$

D1.10 $\frac{3}{5} + \frac{1}{10}$

D1.11 $\frac{5}{6} - \frac{1}{4}$

D1.12 $\frac{2}{7} \cdot \frac{21}{4}$

D1.13 $\frac{3}{4} \div \frac{9}{8}$

D1.14 $1\frac{1}{2} + 2\frac{2}{3}$

D1.15 0.6 as a fraction

D1.16 $0.\overline{4}$ as a fraction

D1.17 $\frac{5}{16}$ as a decimal

D1.18 $8 - 2 \cdot 3^2$

D1.19 $(8 - 2) \cdot 3^2$

D1.20 $24 \div 4 \cdot 2$

D1.21 30% of 70

D1.22 12 is what percent of 48?

D1.23 GCD of 24 and 36

D1.24 LCM of 8 and 12

ANSWERS TO SKILL DRILLS

D1.1 -5. D1.2 -15. D1.3 11. D1.4 42. D1.5 -7. D1.6 1. D1.7 -25. D1.8 18. D1.9 7. D1.10 $\frac{7}{10}$. D1.11 $\frac{7}{12}$. D1.12 $\frac{3}{2}$. D1.13 $\frac{2}{3}$. D1.14 $\frac{25}{6}$. D1.15 $\frac{3}{5}$. D1.16 $\frac{4}{9}$. D1.17 0.3125. D1.18 -10. D1.19 54. D1.20 12. D1.21 21. D1.22 25%. D1.23 12. D1.24 24.

EXERCISES

COMPUTATION

- 1.1** Classify each number as natural, whole, integer, rational, or irrational (list every set it belongs to): $-5, 0, \frac{7}{2}, \sqrt{16}, \sqrt{7}, 0.\overline{12}, \frac{\pi}{2}, 1.75$.
- 1.2** List from least to greatest: $0.3, \frac{1}{3}, -\frac{1}{2}, -0.6, 0.33$.
- 1.3** Evaluate $|-9|, -|4|, |3 - 8|$, and $|-2| - |-5|$.
- 1.4** Find the distance between -7 and 4 , and between -2.5 and -9 .
- 1.5** Compute: $-12 + 7; -6 - 9; 4 - (-11); -3 - (-3)$.
- 1.6** Compute: $(-5)(8); (-7)(-6); (-2)(-3)(-4); (-1)^7$.
- 1.7** Compute $-48 \div 6, (-56) \div (-8)$, and $0 \div (-9)$. Explain what goes wrong with $(-9) \div 0$.
- 1.8** Evaluate $(-3)^2, -3^2, (-2)^3$, and -2^4 .
- 1.9** Reduce to lowest terms: $\frac{42}{56}, \frac{90}{126}, \frac{144}{360}$.
- 1.10** Find the prime factorizations of 84 and 360 , and use them to find the GCD and LCM of the two numbers.
- 1.11** Compute: (a) $\frac{2}{3} + \frac{3}{4}$; (b) $\frac{5}{6} - \frac{7}{10}$; (c) $\frac{1}{4} - \frac{5}{6} + \frac{2}{3}$.
- 1.12** Compute: (a) $\frac{4}{9} \cdot \frac{15}{8}$; (b) $-\frac{3}{5} \div \frac{9}{10}$; (c) $3\frac{1}{2} \div 1\frac{3}{4}$.
- 1.13** Simplify $\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$.
- 1.14** Write as decimals: $\frac{7}{8}, \frac{5}{12}, \frac{2}{11}$.
- 1.15** Write as fractions in lowest terms: $0.45, 0.\overline{7}, 0.\overline{63}, 1.\overline{23}$.
- 1.16** Evaluate: (a) $20 - 3 \cdot 4 + 2$; (b) $(20 - 3) \cdot 4 + 2$; (c) $20 - 3 \cdot (4 + 2)$.
- 1.17** Evaluate $5 + 2[3 - (6 - 10)^2]$.
- 1.18** Evaluate $(-4)^2 - 3(-2)^3 \div 6$.
- 1.19** Evaluate $\frac{2 - 5 \cdot 3}{4 + (-1)^3 \cdot 9}$.
- 1.20** Name the property illustrated: (a) $3 + (x + 4) = (3 + x) + 4$; (b) $5(y - 2) = 5y - 10$; (c) $7 \cdot \frac{1}{7} = 1$; (d) $xy = yx$; (e) $-a + a = 0$.

- 1.21** Use the distributive property to compute mentally: $6 \cdot 99$, $15 \cdot 102$, $8 \cdot 47 + 8 \cdot 53$.
- 1.22** Convert 0.035 to a percent, 135% to a decimal, and $\frac{3}{8}$ to a percent.
- 1.23** (a) What is 15% of 240? (b) 18 is what percent of 72? (c) 45 is 60% of what number?
- 1.24** A town's population grew from 12,500 to 13,750. Find the percent increase.
- 1.25** A \$250 item is marked down 30%, and then an additional 10% is taken off the sale price. Find the final price and the single percent discount that is equivalent to the two.
- 1.26** Solve the proportions: (a) $\frac{x}{12} = \frac{5}{8}$; (b) $\frac{7}{x} = \frac{21}{15}$.
- 1.27** On a map, 1 inch represents 25 miles. Two cities are 3.4 inches apart on the map. How far apart are they?
- 1.28** A car uses 9 gallons of gas to travel 279 miles. At the same rate, how many gallons does it need for 434 miles?

CONCEPTS AND PROOF

- 1.29** Prove that $\sqrt{3}$ is irrational. *Hint:* follow Theorem 1.3; you will need the fact that if 3 divides p^2 , then 3 divides p .
- 1.30** Use the distributive property and the fact $0 \cdot a = 0$ to prove that $(-1)a = -a$, and then that $(-a)(-b) = ab$.
- 1.31** Explain why $\frac{0}{0}$ is undefined, even though $0 \cdot q = 0$ for every number q .
- 1.32** Is the sum of two irrational numbers always irrational? Prove it or give a counterexample.
- 1.33** Show that between any two different rational numbers there is another rational number.
- 1.34** Prove that $|ab| = |a| |b|$ for all real a and b by considering the possible signs of a and b .
- 1.35** A quantity increases by 50% and then decreases by 50%. Find the net percent change, and explain why it is not zero.
- 1.36** Prove that the sum of a rational number and an irrational number is irrational. *Hint:* suppose $r + x = s$ with r, s rational.

CHALLENGE

- 1.37** Write $0.\overline{142857}$ as a fraction in lowest terms.
- 1.38** Show in two different ways that $0.\overline{9} = 1$.
- 1.39** Find the smallest positive integer n such that $\frac{n}{2}$ is a perfect square and $\frac{n}{3}$ is a perfect cube.
- 1.40** A price is raised by $p\%$ and then lowered by $p\%$. The final price is 4% below the original. Find p .

CHAPTER 2

EXPRESSIONS AND EXPONENTS

LEARNING OBJECTIVES

- Evaluate algebraic expressions and formulas, and translate verbal descriptions into expressions.
- Simplify expressions by combining like terms and applying the distributive property.
- State, justify, and apply the laws of exponents, including zero and negative exponents.
- Write numbers in scientific notation and compute with them.

A *variable* is a letter that stands for a number. Using variables, one statement can describe infinitely many calculations at once: $a(b + c) = ab + ac$ says something about every triple of numbers. This chapter develops the basic grammar of algebra: how to read, evaluate, and rewrite expressions without changing their value.

2.1 VARIABLES AND EXPRESSIONS

Definition 2.1 (Expressions and terms). An *algebraic expression* is a combination of numbers and variables using the operations of arithmetic, such as $3x^2 - 2xy + 7$. The parts that are added are its *terms*; here they are $3x^2$, $-2xy$ and 7. The numerical factor of a term is its *coefficient* (3 and -2 here), and a term with no variable is a *constant*.

To *evaluate* an expression, substitute the given values and follow the order of operations. Put each substituted value in parentheses; this prevents most sign errors.

Example 2.2 (Evaluating an expression). Evaluate $3x^2 - 2xy + y$ when $x = -2$ and $y = 3$.

Solution. $3(-2)^2 - 2(-2)(3) + (3) = 3 \cdot 4 + 12 + 3 = 27$. Without the parentheses, $3 - 2^2$ is easy to misread as $3 \cdot (-4)$.

Example 2.3 (Using a formula). The Fahrenheit temperature F that corresponds to a Celsius temperature C is $F = \frac{9}{5}C + 32$. Find F when $C = 25$.

Solution. $F = \frac{9}{5}(25) + 32 = 45 + 32 = 77$. So 25°C is 77°F .

Translating words into expressions is mostly a matter of reading carefully. “The sum of a and b ” is $a + b$, “the product” is ab , “the quotient of a and b ” is $\frac{a}{b}$, and “the difference of a and b ” is $a - b$.

Example 2.4 (Translating). Let n be the number. “Five less than twice a number” is $2n - 5$. “The quotient of a number and 4, increased by 7” is $\frac{n}{4} + 7$. “The square of the sum of a number and 3” is $(n + 3)^2$, while “the sum of the square of a number and 3” is $n^2 + 3$.

COMMON ERROR (“Less than” reverses the order). “Five less than x ” is $x - 5$, not $5 - x$. Test with a number: five less than 12 is $7 = 12 - 5$.

2.2 SIMPLIFYING EXPRESSIONS

Like terms have the same variables raised to the same powers: $4x^2$ and $-x^2$ are like terms, while $4x^2$ and $4x$ are not. The distributive property lets us combine like terms: $4x^2 - x^2 = (4 - 1)x^2 = 3x^2$. Two expressions are *equivalent* if they have the same value for every value of the variables; simplifying replaces an expression with a simpler equivalent one.

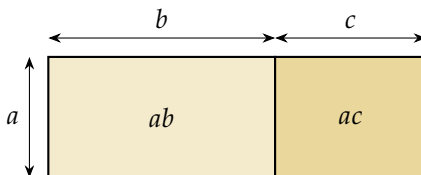


Figure 2.1: The distributive property as area: a rectangle of height a and width $b + c$ has area $a(b + c) = ab + ac$.

Example 2.5 (Combining like terms). Simplify $4x^2 - 3x + 7 - x^2 + 5x - 9$.

Solution. Group like terms: $(4x^2 - x^2) + (-3x + 5x) + (7 - 9) = 3x^2 + 2x - 2$.

Example 2.6 (Distributing). Simplify (a) $3(2a - 5) - 4(a - 1)$ and (b) $2[x - 3(x + 1)] + 5x$.

Solution. (a) $6a - 15 - 4a + 4 = 2a - 11$. Notice $-4 \cdot (-1) = +4$. (b) Work from the inside out: $x - 3(x + 1) = x - 3x - 3 = -2x - 3$, so the expression is $2(-2x - 3) + 5x = -4x - 6 + 5x = x - 6$.

COMMON ERROR (Distributing a minus sign). $-(x - 3) = -x + 3$, not $-x - 3$. A minus sign in front of parentheses is a factor of -1 , and it multiplies *every* term inside.

A simplification can be checked by substituting a number. For (a) above, with $a = 2$: $3(-1) - 4(1) = -7$ and $2(2) - 11 = -7$. The values agree. Agreement for one value does not prove the simplification is right, but disagreement proves it is wrong.

2.3 INTEGER EXPONENTS

Definition 2.7 (Exponents). For a positive integer n , $a^n = a \cdot a \cdots a$ (n factors); a is the *base* and n the *exponent*. For $a \neq 0$ we define

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

Theorem 2.8 (Laws of exponents). For integers m and n , and nonzero a and b :

$$\begin{aligned} a^m a^n &= a^{m+n}, & \frac{a^m}{a^n} &= a^{m-n}, & (a^m)^n &= a^{mn}, \\ (ab)^n &= a^n b^n, & \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n}. \end{aligned}$$

Proof of the first law for positive m, n . $a^m a^n$ is m factors of a followed by n more, which is $m + n$ factors of a in all, so it equals a^{m+n} . The other laws are proved by counting factors in the same way. The definitions of a^0 and a^{-n} are chosen

precisely so that the laws remain true for all integers: for example, $\frac{a^n}{a^n} = 1$, and the quotient law says it is $a^{n-n} = a^0$, so a^0 must be 1. ■

Example 2.9 (Using the laws). (a) $(2x^3y)(-5x^2y^4) = (2)(-5)x^{3+2}y^{1+4} = -10x^5y^5$.

(b) $(3a^2b^{-3})^{-2} = 3^{-2}a^{-4}b^6 = \frac{b^6}{9a^4}$.

(c) $\left(\frac{x^4y^{-2}}{x^{-1}y^3}\right)^2 = \left(x^{4-(-1)}y^{-2-3}\right)^2 = (x^5y^{-5})^2 = \frac{x^{10}}{y^{10}}$.

A negative exponent means reciprocal; it does not make anything negative. So $2^{-3} = \frac{1}{8}$, and a factor with a negative exponent can be moved across a fraction bar by changing the sign of its exponent: $\frac{x^{-2}}{y^{-3}} = \frac{y^3}{x^2}$.

COMMON ERROR (Exponents do not distribute over sums). $(a+b)^2 \neq a^2 + b^2$. With $a = b = 1$: $(1+1)^2 = 4$ but $1^2 + 1^2 = 2$. The law $(ab)^n = a^n b^n$ is for products only. Chapter 7 shows that $(a+b)^2 = a^2 + 2ab + b^2$.

COMMON ERROR (What does the exponent apply to?). $2x^3$ means $2 \cdot x^3$, while $(2x)^3 = 8x^3$. Similarly $-3^0 = -1$ but $(-3)^0 = 1$.

2.4 SCIENTIFIC NOTATION

Very large and very small numbers are written in *scientific notation*: $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer. A positive n moves the decimal point n places right; a negative n moves it left.

Example 2.10 (Converting). $93,000,000 = 9.3 \times 10^7$, and $0.000452 = 4.52 \times 10^{-4}$. In reverse, $6.1 \times 10^{-3} = 0.0061$.

The laws of exponents make computation in scientific notation easy: multiply or divide the leading numbers, add or subtract the exponents, and adjust so the leading number is between 1 and 10.

Example 2.11 (Computing). $(3 \times 10^8)(4 \times 10^{-3}) = 12 \times 10^5 = 1.2 \times 10^6$, and $\frac{6 \times 10^4}{1.5 \times 10^{-2}} = 4 \times 10^6$.

Example 2.12 (A light-year). Light travels about 3.00×10^8 meters per second, and a year has about 3.156×10^7 seconds. How far does light travel in a year?

Solution. $(3.00 \times 10^8)(3.156 \times 10^7) = 9.468 \times 10^{15}$, about 9.47×10^{15} meters.

LOOKING AHEAD. The laws of exponents are extended to fractional exponents such as $a^{1/2} = \sqrt{a}$ in Chapter 11, and to every real exponent in Chapter 12, where the exponent becomes the variable. Expressions built from whole-number powers of a variable are the polynomials of Chapter 7.

2.5 WORKED PROBLEMS

These problems practice evaluating, simplifying, and the laws of exponents. Check each simplification by substituting a number.

PROBLEM 2.1 Height of a thrown ball

A ball thrown upward has height $h = -16t^2 + 48t + 6$ feet after t seconds. Find its height after 1.5 seconds.

Solution. $h = -16(1.5)^2 + 48(1.5) + 6 = -16(2.25) + 72 + 6 = -36 + 78 = 42$ feet. Squaring comes before multiplying by -16 ; computing $(-16 \cdot 1.5)^2$ instead would give a nonsense height of 576 feet before the other terms.

PROBLEM 2.2 Nested grouping

Simplify $5 - 2[3x - (4 - x) + 2(x - 1)]$.

Solution. Inside the brackets: $3x - 4 + x + 2x - 2 = 6x - 6$. Then $5 - 2(6x - 6) = 5 - 12x + 12 = 17 - 12x$. Check with $x = 1$: the original is $5 - 2[3 - 3 + 0] = 5$, and $17 - 12 = 5$.

PROBLEM 2.3 A longer exponent simplification

Simplify $\frac{(2x^2y^{-3})^3(4x^{-1}y)^{-1}}{x^5y^{-9}}$, writing the answer with positive exponents.

Solution. Expand each power: $(2x^2y^{-3})^3 = 8x^6y^{-9}$ and $(4x^{-1}y)^{-1} = \frac{1}{4}xy^{-1}$. Their product is $2x^7y^{-10}$. Dividing by x^5y^{-9} subtracts exponents: $2x^{7-5}y^{-10-(-9)} = 2x^2y^{-1} = \frac{2x^2}{y}$.

PROBLEM 2.4 A perimeter expression

A rectangle's length is 3 more than twice its width w . Write and simplify an expression for its perimeter.

Solution. The length is $2w + 3$, so the perimeter is $2w + 2(2w + 3) = 2w + 4w + 6 = 6w + 6$. For $w = 4$: the rectangle is 4×11 with perimeter 30, and $6(4) + 6 = 30$.

PROBLEM 2.5 How long will the computation take?

A computer performs 2.5×10^9 operations per second. How long does it take to perform 10^{15} operations?

Solution. $\frac{10^{15}}{2.5 \times 10^9} = 0.4 \times 10^6 = 4 \times 10^5$ seconds. A day has 86,400 seconds, so this is $\frac{400,000}{86,400} \approx 4.6$ days.

PROBLEM 2.6 Negative exponents in a fraction

Simplify $\frac{x^{-1} - y^{-1}}{x^{-1} + y^{-1}}$.

Solution. The exponents apply to single terms of a sum, so they cannot simply be moved across the bar. Multiply the numerator and denominator by xy instead: the numerator becomes $y - x$ and the denominator $y + x$. So the expression equals $\frac{y - x}{y + x}$.

PROBLEM 2.7 Estimating a power of two

Estimate 2^{40} in scientific notation without a calculator.

Solution. $2^{10} = 1024 \approx 10^3$, so $2^{40} = (2^{10})^4 \approx (10^3)^4 = 10^{12}$. Since 1024 is 2.4% more than 1000, the estimate is a little low; in fact $2^{40} = 1,099,511,627,776 \approx 1.10 \times 10^{12}$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|--|--|
| D2.1 $3x - 7$ at $x = -2$ | D2.13 $(3y)^2$ |
| D2.2 x^2 at $x = -5$ | D2.14 $(-2x^3)^2$ |
| D2.3 $-x^2$ at $x = -5$ | D2.15 7^0 |
| D2.4 $2a + 3b$ at $a = 4, b = -1$ | D2.16 4^{-2} |
| D2.5 $5y + 3y - y$ | D2.17 $(\frac{2}{5})^{-1}$ |
| D2.6 $4x - 9 - 6x + 2$ | D2.18 $x^{-3}x^5$ |
| D2.7 $3(x + 4)$ | D2.19 $(a^2b)^3$ |
| D2.8 $-(2x - 5)$ | D2.20 $\frac{12x^5}{4x^2}$ |
| D2.9 $2(a - 3) + 4a$ | D2.21 5300 in scientific notation |
| D2.10 $a^2 \cdot a^3$ | D2.22 0.0009 in scientific notation |
| D2.11 $(b^3)^4$ | D2.23 $(2 \times 10^3)(4 \times 10^5)$ |
| D2.12 $\frac{x^{10}}{x^4}$ | D2.24 2.5×10^{-2} in standard form |

ANSWERS TO SKILL DRILLS

D2.1 -13. D2.2 25. D2.3 -25. D2.4 5. D2.5 $7y$. D2.6 $-2x - 7$. D2.7 $3x + 12$. D2.8 $5 - 2x$. D2.9 $6a - 6$. D2.10 a^5 . D2.11 b^{12} . D2.12 x^6 . D2.13 $9y^2$. D2.14 $4x^6$. D2.15 1. D2.16 $\frac{1}{16}$. D2.17 $\frac{5}{2}$. D2.18 x^2 . D2.19 a^6b^3 . D2.20 $3x^3$. D2.21 5.3×10^3 . D2.22 9×10^{-4} . D2.23 8×10^8 . D2.24 0.025.

EXERCISES

COMPUTATION

- 2.1** Evaluate $2x^2 - 5x + 1$ when $x = -3$.
- 2.2** Evaluate $\frac{(a - b)^2}{a + b}$ when $a = 5$ and $b = -1$.
- 2.3** Evaluate $-x^2 + |x - y|$ when $x = -2$ and $y = 4$.
- 2.4** Write an expression for each, using n for the number: (a) seven more than

three times a number; (b) the square of the difference of a number and 2; (c) half of the sum of a number and 9; (d) six less than a number.

2.5 List the terms of $4x^3 - x^2 + \frac{2}{3}x - 9$, and give the coefficient of each variable term.

2.6 The Celsius temperature is $C = \frac{5}{9}(F - 32)$. Find C when $F = 98.6$.

2.7 Simple interest is $I = Prt$. Find I when $P = 2400$, $r = 0.035$ and $t = 3$.

2.8 Simplify $7a - 3b + 2a + 8b$.

2.9 Simplify $5x^2 - 2x + 4 - 3x^2 + 7x - 10$.

2.10 Simplify $4(3y - 2) - (5y + 1)$.

2.11 Simplify $-2(x^2 - 3x + 1) + 3(x^2 + x)$.

2.12 Simplify $3[2t - (t - 4)] - 5t$.

2.13 Simplify $\frac{1}{2}(4x - 6) - \frac{2}{3}(9x + 3)$.

2.14 Simplify $0.3(10p - 20) + 0.05(40p)$.

2.15 Simplify $x^3 \cdot x^5$, $(y^4)^3$, $(2a)^5$, and $\frac{m^9}{m^4}$.

2.16 Simplify $(-3x^2y)(4xy^3)$.

2.17 Simplify $(-2a^3b^2)^3$.

2.18 Simplify $\frac{24p^7q^3}{6p^2q^5}$.

2.19 Evaluate 5^0 , $(-5)^0$, -5^0 , 2^{-3} , $(\frac{1}{3})^{-2}$, and $(-4)^{-2}$.

2.20 Write with positive exponents only: $x^{-4}y^2$, $3a^{-2}$, $(3a)^{-2}$, $\frac{1}{b^{-5}}$.

2.21 Simplify $(2x^{-3}y^4)^{-2}$, using positive exponents in the answer.

2.22 Simplify $(x^2y^{-1})^3(x^{-1}y^2)^2$.

2.23 Simplify $\left(\frac{3m^2n^{-1}}{m^{-2}n^3}\right)^2$.

2.24 Simplify $(a^{-1} + b^{-1})^{-1}$ to a single fraction.

2.25 Write in scientific notation: 93,000,000; 0.00000071; 602,200,000,000,000,000,000,000.

- 2.26** Write in standard form: 4.06×10^5 and 3.2×10^{-4} .
- 2.27** Compute, giving the answer in scientific notation: (a) $(6 \times 10^4)(5 \times 10^{-9})$;
(b) $\frac{8.4 \times 10^6}{2.1 \times 10^{-2}}$.
- 2.28** The mass of the Earth is 5.97×10^{24} kg and that of the Moon is 7.35×10^{22} kg. About how many times as massive as the Moon is the Earth?
- 2.29** Light travels 3.00×10^8 m/s, and the Sun is 1.50×10^{11} m from the Earth. How many seconds does sunlight take to reach the Earth? How many minutes?

CONCEPTS AND PROOF

- 2.30** Prove that $a^m a^n = a^{m+n}$ and $\frac{a^m}{a^n} = a^{m-n}$ for positive integers $m > n$, by counting factors.
- 2.31** Explain why a^0 must be defined as 1 (for $a \neq 0$) if the quotient law is to hold when $m = n$.
- 2.32** Show that the definition $a^{-n} = \frac{1}{a^n}$ makes the product law true in the example $a^5 \cdot a^{-2} = a^3$.
- 2.33** Is $(a + b)^2 = a^2 + b^2$ for all numbers? Give a counterexample, and draw a square of side $a + b$ to explain what is missing.
- 2.34** Use the commutative and associative properties to prove that $(ab)^3 = a^3 b^3$.
- 2.35** Explain why $-x^2$ is never positive, while $(-x)^2$ is never negative.
- 2.36** Are $2(x + 3) - 2x$ and 6 equivalent expressions? Are $\frac{x}{x}$ and 1? Explain.

CHALLENGE

- 2.37** Simplify $\frac{2^{n+2} - 2^n}{3 \cdot 2^n}$.
- 2.38** Which is larger, 2^{100} or 3^{60} ? Answer without a calculator.
- 2.39** How many digits does $4^{10} \cdot 5^{18}$ have?
- 2.40** If $x + \frac{1}{x} = 3$, find $x^2 + \frac{1}{x^2}$.

CHAPTER 3

LINEAR EQUATIONS AND INEQUALITIES

LEARNING OBJECTIVES

- Solve linear equations, including equations with fractions and decimals, and recognize identities and contradictions.
- Solve formulas for a specified variable.
- Translate word problems into equations and solve them.
- Solve linear and compound inequalities, and write their solutions in interval notation.
- Solve equations and inequalities involving absolute value.

An *equation* states that two expressions are equal, and a *solution* is a value of the variable that makes the statement true. Solving an equation means finding all of its solutions. The method is to replace the equation, step by step, with simpler equations that have exactly the same solutions, until the solution can be read off.

3.1 LINEAR EQUATIONS

Theorem 3.1 (Properties of equality). For real numbers a , b , c :

- (1) $a = b$ if and only if $a + c = b + c$.
- (2) If $c \neq 0$, then $a = b$ if and only if $ac = bc$.

So adding the same number to both sides, or multiplying both sides by the same nonzero number, gives an equivalent equation, one with exactly the same solutions.

Definition 3.2 (Linear equation). A *linear equation* in x is one that can be written as $ax + b = 0$ with $a \neq 0$. It has exactly one solution, $x = -\frac{b}{a}$.

Example 3.3 (Variables on both sides). Solve $3(2x - 1) - 4 = 2(x + 5) + 1$.

Solution. Simplify each side: $6x - 7 = 2x + 11$. Subtract $2x$ and add 7: $4x = 18$. Divide by 4: $x = \frac{9}{2}$. *Check:* the left side is $3(8) - 4 = 20$ and the right side is $2\left(\frac{19}{2}\right) + 1 = 20$.

When an equation contains fractions, multiply both sides by the LCD of all the fractions first. This *clears the fractions* and leaves an equation with integer coefficients.

Example 3.4 (Clearing fractions). Solve $\frac{x}{3} - \frac{x-1}{4} = 2$.

Solution. Multiply both sides by 12: $4x - 3(x - 1) = 24$, so $4x - 3x + 3 = 24$ and $x = 21$. *Check:* $7 - 5 = 2$.

COMMON ERROR (Multiply *every* term). When clearing fractions, every term on both sides is multiplied, including terms without denominators. In the example above, the 2 becomes 24. Also note the parentheses around $x - 1$: the minus sign in front of the fraction applies to the whole numerator.

Not every equation that looks linear has exactly one solution. If the variable disappears, the equation is either always true or never true.

Example 3.5 (Identities and contradictions). $2(x + 3) = 2x + 6$ simplifies to $6 = 6$, which is true for every x : it is an *identity*, and every real number is a solution. $2(x + 3) = 2x + 5$ simplifies to $6 = 5$, which is never true: it is a *contradiction* and has no solution.

3.2 FORMULAS

A formula relates several variables. Solving a formula for one of them uses exactly the same steps as solving an equation, treating the other letters as numbers.

Example 3.6 (Solving a formula). The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$. Solve for b_1 .

Solution. Multiply by 2: $2A = h(b_1 + b_2)$. Divide by h : $\frac{2A}{h} = b_1 + b_2$. Subtract b_2 : $b_1 = \frac{2A}{h} - b_2$.

Example 3.7 (Collecting the variable). Solve $ax + 3 = cx - 5$ for x .

Solution. Collect the x -terms on one side: $ax - cx = -8$. Factor out x : $(a - c)x = -8$. If $a \neq c$, then $x = \frac{-8}{a - c} = \frac{8}{c - a}$. If $a = c$, the equation becomes $3 = -5$ and has no solution.

COMMON ERROR (Dividing by an expression). Dividing by $a - c$ is allowed only when $a - c \neq 0$. When the divisor contains a variable, state the condition, as in the example above.

3.3 APPLICATIONS

Definition 3.8 (A strategy for word problems). (1) Read the problem and identify the unknown quantity. Assign it a variable, and write down what the variable means, with units.

- (2) Express the other quantities in terms of the variable.
- (3) Write an equation that says the same thing as the problem.
- (4) Solve the equation, and answer the question that was asked.
- (5) Check the answer against the original wording, not against your equation.

Example 3.9 (Consecutive integers). The sum of three consecutive odd integers is 87. Find them.

Solution. Let n be the smallest. Consecutive odd integers differ by 2, so the others are $n + 2$ and $n + 4$. Then $3n + 6 = 87$, so $n = 27$. The integers are 27, 29 and 31, and $27 + 29 + 31 = 87$.

Example 3.10 (Motion). Two cars leave the same point at the same time, traveling in opposite directions at 55 mph and 65 mph. When are they 300 miles apart?

Solution. Distance is rate times time. After t hours the cars have traveled

$55t$ and $65t$ miles, so they are $120t$ miles apart. Then $120t = 300$ and $t = 2.5$ hours.

Example 3.11 (Investment). \$10,000 is invested, part at 4% and the rest at 6% simple interest. The total interest after one year is \$520. How much is invested at each rate?

Solution. Let x be the amount at 4%; then $10,000 - x$ is at 6%. The interest equation is $0.04x + 0.06(10,000 - x) = 520$, so $600 - 0.02x = 520$ and $x = 4000$. So \$4000 is at 4% and \$6000 at 6%. *Check:* $160 + 360 = 520$.

3.4 LINEAR INEQUALITIES

An inequality such as $4 - 3x \geq 16$ usually has infinitely many solutions, and solving it means describing all of them. The rules are those of equations, with one important change.

Theorem 3.12 (Properties of inequalities). For real numbers a, b, c :

- (1) If $a < b$, then $a + c < b + c$.
- (2) If $a < b$ and $c > 0$, then $ac < bc$.
- (3) If $a < b$ and $c < 0$, then $ac > bc$: multiplying by a negative number reverses the inequality.

The same rules hold for $\leq$, $>$ and $\geq$, and for division, since dividing by c is multiplying by $\frac{1}{c}$.

The reversal is visible on the number line: $2 < 5$, but $-2 > -5$, because multiplying by -1 reflects every point across 0 and so reverses left and right. Solution sets are written in *interval notation*: a square bracket includes an endpoint and a parenthesis excludes it.

Inequality	Interval	Graph
$a < x < b$	(a, b)	open dots at a and b , shaded between
$a \leq x < b$	$[a, b)$	closed dot at a , open dot at b
$x \geq a$	$[a, \infty)$	closed dot at a , shaded to the right
$x < b$	$(-\infty, b)$	open dot at b , shaded to the left

Example 3.13 (Reversing the inequality). Solve $4 - 3x \geq 16$.

Solution. Subtract 4: $-3x \geq 12$. Divide by -3 , reversing the inequality: $x \leq -4$. The solution set is $(-\infty, -4]$. Check: $x = -5$ gives $19 \geq 16$, true, and $x = 0$ gives $4 \geq 16$, false.

A *compound inequality* combines two inequalities. One joined by “and” is satisfied when both parts are true; one joined by “or” when at least one part is true. A double inequality $a < x < b$ means $a < x$ and $x < b$, and can be solved in one pass by operating on all three parts at once.

Example 3.14 (A double inequality). Solve $-1 < 2x + 3 \leq 9$.

Solution. Subtract 3 from all three parts: $-4 < 2x \leq 6$. Divide by 2: $-2 < x \leq 3$. The solution set is $(-2, 3]$, shown in Fig. 3.1.



Figure 3.1: An “and” inequality gives one interval; an “or” inequality may give two. A closed dot marks an included endpoint and an open dot an excluded one.

3.5 ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Since $|X|$ is the distance from X to 0, the statement $|X| = a$ (with $a > 0$) means X is a units from 0, so $X = a$ or $X = -a$. Distances lead similarly to the inequality rules.

Theorem 3.15 (Absolute value equations and inequalities). Let $a > 0$ and let X be any expression.

- (1) $|X| = a$ if and only if $X = a$ or $X = -a$.
- (2) $|X| < a$ if and only if $-a < X < a$.
- (3) $|X| > a$ if and only if $X < -a$ or $X > a$.

The same holds with $\leq$ and $\geq$. In particular, $|x - c| < a$ says that x is within a units of c .

Example 3.16 (An absolute value equation). Solve $|2x - 3| = 7$.

Solution. $2x - 3 = 7$ gives $x = 5$, and $2x - 3 = -7$ gives $x = -2$. Both check: $|7| = 7$ and $|-7| = 7$.

Example 3.17 (Absolute value inequalities). Solve (a) $|3x + 1| \leq 8$ and (b) $|x - 4| > 2$.

Solution. (a) $-8 \leq 3x + 1 \leq 8$, so $-9 \leq 3x \leq 7$ and $-3 \leq x \leq \frac{7}{3}$: the interval $[-3, \frac{7}{3}]$. (b) $x - 4 < -2$ or $x - 4 > 2$, so $x < 2$ or $x > 6$: the set $(-\infty, 2) \cup (6, \infty)$. These are the points more than 2 units from 4.

COMMON ERROR (Negative right-hand sides). An absolute value is never negative. So $|x - 5| = -2$ and $|x - 5| < -2$ have no solution, while $|x - 5| > -2$ is true for every x . Applying the rules of Theorem 3.15 with $a < 0$ gives wrong answers.

LOOKING AHEAD. Absolute value inequalities are the language of calculus: the definition of a limit says that $|f(x) - L|$ can be made smaller than any positive number by keeping $|x - a|$ small enough. In Chapter 6, linear equations in two variables are solved together, and in Chapter 9 the equation $ax^2 + bx + c = 0$ is solved for every case.

3.6 WORKED PROBLEMS

These problems combine equation solving with modeling. In each word problem, write down what the variable means before writing the equation.

PROBLEM 3.1 An equation with two fractions

Solve $\frac{x + 3}{4} - \frac{2x - 1}{6} = \frac{1}{3}$.

Solution. The LCD is 12. Multiplying every term: $3(x + 3) - 2(2x - 1) = 4$, so $3x + 9 - 4x + 2 = 4$. Then $-x + 11 = 4$ and $x = 7$. Check: $\frac{10}{4} - \frac{13}{6} = \frac{30-26}{12} = \frac{4}{12} = \frac{1}{3}$.

PROBLEM 3.2 When do the scales agree?

At what temperature do the Celsius and Fahrenheit scales show the same number?

Solution. We need $F = C$ in $F = \frac{9}{5}C + 32$. Substituting C for F : $C = \frac{9}{5}C + 32$, so $-\frac{4}{5}C = 32$ and $C = -40$. So -40°C is the same as -40°F .

PROBLEM 3.3 Catching up

A car leaves a town at 9:00 driving 48 mph. A second car leaves the same town at 10:00 on the same road, driving 64 mph. When does it catch the first car, and how far from town?

Solution. Let t be the hours after 9:00. The first car has gone $48t$ miles and the second, which has driven for $t - 1$ hours, $64(t - 1)$ miles. They meet when $48t = 64(t - 1)$, so $16t = 64$ and $t = 4$. The second car catches up at 1:00, $48 \cdot 4 = 192$ miles from town.

PROBLEM 3.4 Strengthening a mixture

A radiator holds 12 liters of 30% antifreeze. How much pure antifreeze must be added to make a 50% mixture?

Solution. Track the amount of pure antifreeze. There are $0.30(12) = 3.6$ liters now. Adding x liters of pure antifreeze gives $3.6 + x$ liters in a total of $12 + x$ liters, which should be 50%: $3.6 + x = 0.5(12 + x)$. So $0.5x = 2.4$ and $x = 4.8$ liters.

PROBLEM 3.5 Breaking even

A small business has fixed monthly costs of \$1200. Each item costs \$8 to make and sells for \$20. How many items must it sell in a month to make a profit?

Solution. Profit is revenue minus cost: $20n - (1200 + 8n) = 12n - 1200$. A profit means $12n - 1200 > 0$, so $n > 100$. The business must sell at least 101 items. At exactly 100 items it breaks even.

PROBLEM 3.6 A manufacturing tolerance

A machined part should be 5.00 cm long, and parts within 0.02 cm of that length are accepted. Write the accepted lengths as an absolute value in-

equality and as an interval.

Solution. “Within 0.02 of 5.00” is $|L - 5| \leq 0.02$. By Theorem 3.15, $-0.02 \leq L - 5 \leq 0.02$, so $4.98 \leq L \leq 5.02$: the interval $[4.98, 5.02]$.

PROBLEM 3.7 A variable on both sides of an absolute value equation

Solve $|x - 3| = 2x$.

Solution. The right side must be nonnegative, since it equals an absolute value, so any solution has $x \geq 0$. The two cases are $x - 3 = 2x$, giving $x = -3$, and $x - 3 = -2x$, giving $x = 1$. The first fails the condition $x \geq 0$ (and indeed $|-6| \neq -6$). The only solution is $x = 1$: $|-2| = 2$.

PROBLEM 3.8 An inequality with fractions

Solve $\frac{1 - 2x}{3} > \frac{x + 4}{2} - 1$.

Solution. Multiply by 6, which is positive, so the inequality keeps its direction: $2(1 - 2x) > 3(x + 4) - 6$, that is, $2 - 4x > 3x + 6$. Then $-7x > 4$, and dividing by -7 reverses it: $x < -\frac{4}{7}$. The solution set is $(-\infty, -\frac{4}{7})$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D3.1 $x + 7 = 3$

D3.2 $4x = -20$

D3.3 $\frac{x}{3} = 6$

D3.4 $2x - 5 = 9$

D3.5 $7 - x = 10$

D3.6 $3x + 2 = x - 6$

D3.7 $2(x - 1) = 10$

D3.8 $\frac{x}{2} + 1 = 4$

D3.9 $0.5x = 3.5$

D3.10 $5 - 2(x + 1) = 1$

D3.11 Solve $d = rt$ for t .

D3.12 Solve $y = mx + b$ for m .

D3.13 Solve $2x + 3y = 6$ for y .

D3.14 $x + 4 > 1$

D3.15 $-3x \leq 12$

D3.16 $2x - 1 \geq 5$

D3.17 $-x > 2$

D3.18 $0 < x - 1 < 4$

D3.19 $-2 \leq 3x + 1 \leq 7$

D3.20 $|x| = 6$

D3.21 $|x - 2| = 5$

D3.23 $|x + 1| \geq 2$

D3.22 $|x| < 3$

D3.24 $|2x| = -4$

ANSWERS TO SKILL DRILLS

D3.1 $x = -4$. D3.2 $x = -5$. D3.3 $x = 18$. D3.4 $x = 7$. D3.5 $x = -3$. D3.6 $x = -4$. D3.7 $x = 6$. D3.8 $x = 6$. D3.9 $x = 7$. D3.10 $x = 1$. D3.11 $t = \frac{d}{r}$.
D3.12 $m = \frac{y-b}{x}$. D3.13 $y = 2 - \frac{2}{3}x$. D3.14 $x > -3$. D3.15 $x \geq -4$. D3.16 $x \geq 3$. D3.17 $x < -2$. D3.18 $1 < x < 5$. D3.19 $-1 \leq x \leq 2$. D3.20 $x = \pm 6$.
D3.21 $x = -3$ or $x = 7$. D3.22 $-3 < x < 3$. D3.23 $x \leq -3$ or $x \geq 1$. D3.24 No solution.

EXERCISES

COMPUTATION

3.1 Solve $7x - 4 = 24$.

3.2 Solve $3 - 2x = 11$.

3.3 Solve $5(x - 2) = 3x + 4$.

3.4 Solve $4(2y + 1) - 3(y - 2) = 25$.

3.5 Solve $0.2x + 1.5 = 0.5x - 0.3$.

3.6 Solve $\frac{x}{2} + \frac{x}{3} = 10$.

3.7 Solve $\frac{2x - 1}{5} - \frac{x + 2}{3} = 1$.

3.8 Classify each as an identity, a contradiction, or a conditional equation, and solve: (a) $3(x - 1) + 3 = 3x$; (b) $2x + 5 = 2(x + 2)$; (c) $4x - 1 = 3x$.

3.9 Solve $V = \pi r^2 h$ for h .

3.10 Solve $3x - 4y = 12$ for y .

3.11 Solve $C = \frac{5}{9}(F - 32)$ for F .

3.12 Solve $A = P + Prt$ for t .

3.13 Solve $ax - b = cx + d$ for x , assuming $a \neq c$.

3.14 Solve $S = \frac{a}{1 - r}$ for r .

- 3.15** The sum of three consecutive integers is 141. Find them.
- 3.16** One number is 8 more than three times another, and their sum is 52. Find the numbers.
- 3.17** A rectangle has perimeter 86 cm, and its length is 7 cm more than its width. Find its dimensions.
- 3.18** In a triangle, the second angle is twice the first, and the third is 20° more than the first. Find the angles.
- 3.19** Two trains leave a station at the same time in opposite directions. One travels 15 mph faster than the other, and after 3 hours they are 345 miles apart. Find their speeds.
- 3.20** \$8000 is invested, part at 3% and the rest at 5% simple interest, earning \$340 in one year. How much is invested at each rate?
- 3.21** How many liters of a 10% acid solution must be added to 20 liters of a 40% solution to make a 25% solution?
- 3.22** A jar holds 30 coins, all dimes and quarters, worth \$5.40. How many of each are there?
- 3.23** Solve and write in interval notation: $3x - 5 < 7$.
- 3.24** Solve: $-2x + 1 \geq 9$.
- 3.25** Solve: $5(x - 1) > 2(x + 2)$.
- 3.26** Solve: $\frac{x}{4} - 1 \leq \frac{x + 2}{6}$.
- 3.27** Solve: $-3 \leq 2x - 1 < 5$.
- 3.28** Solve: $1 < \frac{4 - x}{2} \leq 3$.
- 3.29** Solve: $x + 2 < 0$ or $3x \geq 9$.
- 3.30** Solve $|x - 3| = 8$.
- 3.31** Solve $|4x + 2| = 10$.
- 3.32** Solve $|2x - 1| = |x + 4|$.
- 3.33** Solve $|x + 2| < 5$.
- 3.34** Solve $|3 - 2x| \geq 7$.

3.35 Solve $\left|\frac{x}{2} - 1\right| \leq 3$.

3.36 A phone plan costs \$25 per month plus \$0.10 per text message. How many messages can be sent if the bill is to be at most \$40?

3.37 A student scored 78, 85 and 91 on three tests. What score on the fourth test gives an average of at least 85?

CONCEPTS AND PROOF

3.38 Prove that if $a < b$ and $c < 0$, then $ac > bc$. *Hint: $a < b$ means $b - a > 0$, and the product of a positive and a negative number is negative.*

3.39 Why is it a mistake to divide both sides of $x(x - 2) = 3x$ by x ? Which solution is lost?

3.40 Prove that if $a < b$ and $c < d$, then $a + c < b + d$.

3.41 Write an absolute value inequality whose solution set is the interval $(2, 10)$.

3.42 Explain why $|x - 5| = -2$ has no solution while every real number satisfies $|x - 5| > -2$.

3.43 A student solves $\frac{2}{x} < 1$ by multiplying both sides by x to get $x > 2$. Find a value of x that satisfies the original inequality but not the student's answer, and explain the error.

CHALLENGE

3.44 Solve $|x - 1| + |x + 2| = 5$. *Hint: consider $x \geq 1$, $-2 \leq x < 1$ and $x < -2$ separately.*

3.45 At what time between 3:00 and 4:00 do the hour and minute hands of a clock first coincide?

3.46 Let $a < b$. Solve $|x - a| < |x - b|$, and interpret the answer as a statement about distances.



PART II

LINEAR RELATIONSHIPS

The coordinate plane and lines, functions and their graphs, and systems of linear equations and inequalities.

CHAPTER 4

GRAPHS AND LINES

LEARNING OBJECTIVES

- Plot points, and use the distance and midpoint formulas.
- Graph equations by plotting points and finding intercepts.
- Compute and interpret the slope of a line.
- Write equations of lines in slope-intercept, point-slope, and standard form.
- Use slopes to recognize parallel and perpendicular lines.
- Build and interpret linear models of real situations.

In the seventeenth century, Descartes and Fermat joined algebra to geometry: points became pairs of numbers, and curves became equations. The result, *coordinate geometry*, lets a picture check an equation and an equation describe a picture. This chapter applies the idea to the simplest curves, straight lines, which are the graphs of linear equations.

4.1 THE COORDINATE PLANE

Draw a horizontal number line (the *x-axis*) and a vertical one (the *y-axis*) crossing at their zeros, the *origin*. Each point *P* in the plane is then described by an *ordered pair* (a, b) : *a* is its horizontal position and *b* its vertical position. The axes divide the plane into four *quadrants*, numbered I to IV counterclockwise from the upper right.

Theorem 4.1 (Distance and midpoint formulas). *The distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

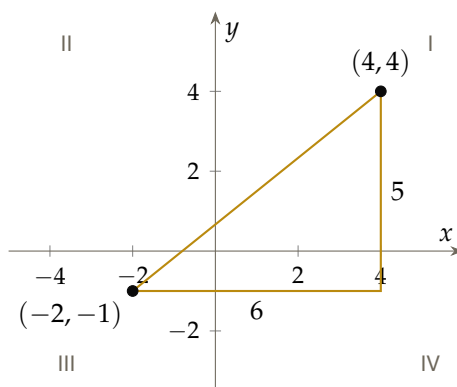


Figure 4.1: The distance between two points is the hypotenuse of a right triangle whose legs are the horizontal and vertical separations; here $d = \sqrt{6^2 + 5^2}$.

and the midpoint of the segment P_1P_2 is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Proof of the distance formula. The points P_1 , P_2 and (x_2, y_1) form a right triangle (Fig. 4.1) with horizontal leg $|x_2 - x_1|$ and vertical leg $|y_2 - y_1|$. By the Pythagorean Theorem, $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Squaring removes the absolute value signs. ■

Example 4.2 (Distance and midpoint). For $(-2, -1)$ and $(4, 4)$: $d = \sqrt{6^2 + 5^2} = \sqrt{61} \approx 7.81$, and the midpoint is $\left(\frac{-2+4}{2}, \frac{-1+4}{2}\right) = (1, 1.5)$.

4.2 GRAPHS OF EQUATIONS

The *graph* of an equation in x and y is the set of all points (x, y) that satisfy it. To sketch a graph, make a table of solution points and connect them; for anything other than a line, plot enough points to see the shape. The points where a graph meets the axes are especially useful.

Definition 4.3 (Intercepts). An *x-intercept* is a point where the graph meets the x -axis; find it by setting $y = 0$. A *y-intercept* is a point where the graph meets the y -axis; find it by setting $x = 0$.

Example 4.4 (Intercepts). Find the intercepts of (a) $3x - 4y = 12$ and (b) $y = x^2 - 4$.

Solution. (a) $y = 0$ gives $x = 4$, and $x = 0$ gives $y = -3$: the intercepts are $(4, 0)$ and $(0, -3)$, and the line through them is the graph. (b) $y = 0$ gives $x^2 = 4$, so $x = \pm 2$; $x = 0$ gives $y = -4$. The graph is a U-shaped curve, a *parabola*, through $(-2, 0)$, $(0, -4)$ and $(2, 0)$, studied in Chapter 9.

4.3 SLOPE

Definition 4.5 (Slope). The *slope* of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A vertical line ($x_1 = x_2$) has no slope; its slope is *undefined*.

The slope does not depend on which two points of the line are used: two such slope triangles are similar, so their ratios of rise to run are equal. Slope measures steepness and direction. A line with positive slope rises to the right; with negative slope it falls; with slope 0 it is horizontal. A slope of m means that y changes by m units for each unit increase in x : slope is a *rate of change*.

Example 4.6 (Computing slopes). Through $(2, 3)$ and $(6, 11)$: $m = \frac{11-3}{6-2} = 2$. Through $(-1, 4)$ and $(3, -2)$: $m = \frac{-2-4}{3-(-1)} = -\frac{3}{2}$. Through $(5, 2)$ and $(5, -3)$: the run is 0, so the line is vertical and the slope is undefined.

COMMON ERROR (Keep the order consistent). In $\frac{y_2 - y_1}{x_2 - x_1}$, the point subtracted first on top must be subtracted first on the bottom. Computing $\frac{y_2 - y_1}{x_1 - x_2}$ gives the wrong sign.

4.4 EQUATIONS OF LINES

If a line has slope m and passes through (x_1, y_1) , then any other point (x, y) on it satisfies $\frac{y - y_1}{x - x_1} = m$. Clearing the denominator gives the first form below.

Theorem 4.7 (Forms of the equation of a line).

Point-slope form	$y - y_1 = m(x - x_1)$ (slope m , through (x_1, y_1))
Slope-intercept form	$y = mx + b$ (slope m , y -intercept $(0, b)$)
Standard form	$Ax + By = C$ (A, B not both 0)
Horizontal line	$y = b$ (slope 0)
Vertical line	$x = a$ (undefined slope)

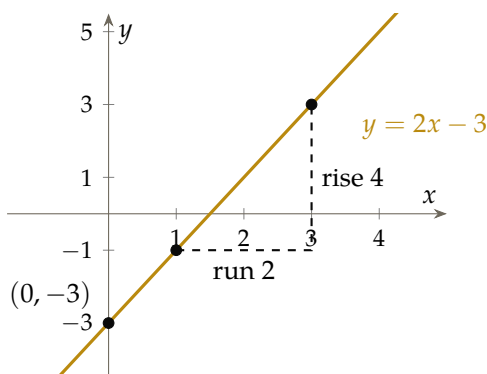


Figure 4.2: The line $y = 2x - 3$ has y -intercept -3 and slope $\frac{4}{2} = 2$.

Example 4.8 (A line through two points). Find the line through $(2, 5)$ and $(-1, -4)$.

Solution. The slope is $\frac{-4-5}{-1-2} = 3$. Point-slope form with $(2, 5)$: $y - 5 = 3(x - 2)$. Solving for y : $y = 3x - 1$. Check the other point: $3(-1) - 1 = -4$.

Example 4.9 (Reading slope and intercept). Find the slope and y -intercept of $2x + 5y = 10$, and graph the line.

Solution. Solve for y : $5y = -2x + 10$, so $y = -\frac{2}{5}x + 2$. The slope is $-\frac{2}{5}$ and the y -intercept is $(0, 2)$. To graph, start at $(0, 2)$ and move 5 right and 2 down to $(5, 0)$, then draw the line through the two points.

4.5 PARALLEL AND PERPENDICULAR LINES

Theorem 4.10 (Parallel and perpendicular lines). *Two nonvertical lines are parallel if and only if they have the same slope. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, $m_2 = -\frac{1}{m_1}$: the slopes are negative reciprocals.*

Parallel lines have the same steepness, so they have equal slopes. For perpendicular lines, rotating a slope triangle with run 1 and rise m by a quarter turn gives one with run $-m$ and rise 1, whose slope is $-\frac{1}{m}$. Exercise 37 gives an algebraic proof.

Example 4.11 (A perpendicular line). Find the line through $(6, -1)$ perpendicular to $y = 3x + 1$.

Solution. The given slope is 3, so the perpendicular slope is $-\frac{1}{3}$. Then $y + 1 = -\frac{1}{3}(x - 6)$, so $y = -\frac{1}{3}x + 1$.

Example 4.12 (Classifying a pair of lines). Are $4x - 6y = 5$ and $3x + 2y = 1$ parallel, perpendicular, or neither?

Solution. Solving for y , the slopes are $\frac{4}{6} = \frac{2}{3}$ and $-\frac{3}{2}$. Their product is -1 , so the lines are perpendicular.

4.6 LINEAR MODELS

A quantity that changes at a constant rate is modeled by a linear equation. In $y = mx + b$, the slope m is the rate of change, in units of y per unit of x , and the intercept b is the value of y when $x = 0$, often the starting value.

Example 4.13 (A service charge). A plumber charges a \$65 visit fee plus \$40 per hour. Write the cost C of a job lasting h hours, and find the cost of a 3.5-hour job.

Solution. $C = 65 + 40h$. The slope 40 is dollars per hour, and the intercept 65 is the cost of a visit with no work. For $h = 3.5$, $C = 65 + 140 = 205$ dollars.

Example 4.14 (A model from two data points). A car is worth \$18,000 when it is 3 years old and \$10,000 when it is 7 years old. Assuming its value decreases linearly, find a formula for its value, its value when new, and when it will be worth nothing.

Solution. The points $(3, 18000)$ and $(7, 10000)$ give slope $\frac{10000-18000}{7-3} = -2000$ dollars per year. Then $V - 18000 = -2000(t - 3)$, so $V = 24000 - 2000t$. When new ($t = 0$) it was worth \$24,000, and $V = 0$ when $t = 12$ years. A model like this is useful only within a reasonable range: real cars do not lose value at a constant rate forever.

LOOKING AHEAD. Slope is the central idea of calculus. For a curve, the slope changes from point to point, and the *derivative* of Volume III is the slope of the line that best fits the curve at a single point. Every linear model in this section is also a *linear function*, the first family of functions studied in Chapter 5.

4.7 WORKED PROBLEMS

These problems use slopes, distances, and equations of lines together. A quick sketch makes most of them easier.

PROBLEM 4.1 A right triangle from slopes

Show that $A(1, 2)$, $B(5, 4)$ and $C(3, 8)$ form a right triangle, and find its area.

Solution. The slope of AB is $\frac{4-2}{5-1} = \frac{1}{2}$ and the slope of BC is $\frac{8-4}{3-5} = -2$. Their product is -1 , so $AB \perp BC$ and the right angle is at B . The legs have lengths $AB = \sqrt{16+4} = \sqrt{20}$ and $BC = \sqrt{4+16} = \sqrt{20}$, so the area is $\frac{1}{2}\sqrt{20}\sqrt{20} = 10$.

PROBLEM 4.2 Parallel to a line through two points

Find the line through $(4, -1)$ parallel to the line through $(-2, 5)$ and $(1, -4)$.

Solution. The given line has slope $\frac{-4-5}{1-(-2)} = -3$, and a parallel line has the same slope. So $y + 1 = -3(x - 4)$, which gives $y = -3x + 11$.

PROBLEM 4.3 Modeling growth

A gym had 1200 members in 2019 and 1950 in 2024. Assuming linear growth, find a model, interpret its slope, and predict the membership in 2027.

Solution. Let t be years after 2019. The points $(0, 1200)$ and $(5, 1950)$ give slope $\frac{750}{5} = 150$, so $M = 1200 + 150t$. The gym gains 150 members per year. In 2027, $t = 8$ and $M = 1200 + 1200 = 2400$ members. The prediction assumes the trend continues, which is a guess, not a fact.

PROBLEM 4.4 A budget limit

A rental truck costs \$49.95 per day plus \$0.79 per mile. With a budget of \$150 for one day, how many miles can be driven?

Solution. The cost is $49.95 + 0.79m \leq 150$, so $0.79m \leq 100.05$ and $m \leq 126.6$. Rounding *down*, since rounding up would exceed the budget, the answer is 126 miles.

PROBLEM 4.5 The center of a circle

Find the center and radius of the circle through $(0, 0)$, $(8, 0)$ and $(0, 6)$.

Solution. The three points form a right triangle with the right angle at the origin. A right triangle's hypotenuse is a diameter of the circle through its vertices, so the center is the midpoint of the segment from $(8, 0)$ to $(0, 6)$, which is $(4, 3)$. The radius is the distance from $(4, 3)$ to $(0, 0)$: $\sqrt{16 + 9} = 5$. It is the same distance to each of the other two points, as a check.

PROBLEM 4.6 A triangle of given area

For which values of c does the line $2x + 3y = c$ form a triangle of area 12 with the coordinate axes?

Solution. The intercepts are $x = \frac{c}{2}$ and $y = \frac{c}{3}$. The triangle has legs $|\frac{c}{2}|$ and $|\frac{c}{3}|$, so its area is $\frac{1}{2} \cdot \frac{c^2}{6} = \frac{c^2}{12}$. Setting $\frac{c^2}{12} = 12$ gives $c^2 = 144$, so $c = 12$ or $c = -12$.

PROBLEM 4.7 Temperature and altitude

Air temperature drops about 6.5°C for every 1000 m of altitude. If it is 20°C at sea level, find a model, the temperature at 3000 m, and the altitude where

the temperature reaches freezing.

Solution. The rate is -6.5 degrees per 1000 m, or -0.0065 degrees per meter, so $T = 20 - 0.0065h$. At $h = 3000$, $T = 20 - 19.5 = 0.5^\circ\text{C}$. Freezing means $T = 0$: $h = \frac{20}{0.0065} \approx 3077$ m.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|---|--|
| D4.1 Quadrant of $(-3, 5)$ | D4.15 Line through $(1, 1)$ and $(2, 3)$ |
| D4.2 Distance from $(0, 0)$ to $(3, 4)$ | D4.16 Line with slope $\frac{1}{2}$ through $(4, 0)$ |
| D4.3 Distance from $(1, 1)$ to $(4, 5)$ | D4.17 Horizontal line through $(5, -2)$ |
| D4.4 Midpoint of $(2, 6)$ and $(8, 0)$ | D4.18 Vertical line through $(5, -2)$ |
| D4.5 Slope through $(1, 2)$ and $(3, 8)$ | D4.19 Slope of a line parallel to
$y = -3x + 2$ |
| D4.6 Slope through $(0, 5)$ and $(4, 3)$ | D4.20 Slope of a line perpendicular to
$y = -3x + 2$ |
| D4.7 Slope through $(2, 7)$ and $(2, 1)$ | D4.21 Slope of a line perpendicular to
$y = \frac{2}{5}x$ |
| D4.8 Slope of $y = 4x - 1$ | D4.22 $y = 2x + 1$ and $y = -\frac{1}{2}x + 3$:
parallel, perpendicular, or
neither? |
| D4.9 y -intercept of $y = -2x + 7$ | D4.23 $y = x$ and $y = x + 4$: parallel,
perpendicular, or neither? |
| D4.10 Slope of $3x + y = 5$ | D4.24 Is $(3, 5)$ on $y = 2x - 1$? |
| D4.11 Slope of $2x - 3y = 6$ | |
| D4.12 x -intercept of $y = 2x - 8$ | |
| D4.13 Line with slope 2 and
y -intercept -3 | |
| D4.14 Line with slope -1 through
$(0, 4)$ | |

ANSWERS TO SKILL DRILLS

D4.1 II. D4.2 5. D4.3 5. D4.4 $(5, 3)$. D4.5 3. D4.6 $-\frac{1}{2}$. D4.7 Undefined.
D4.8 4. D4.9 7. D4.10 -3 . D4.11 $\frac{2}{3}$. D4.12 4. D4.13 $y = 2x - 3$. D4.14
 $y = -x + 4$. D4.15 $y = 2x - 1$. D4.16 $y = \frac{1}{2}x - 2$. D4.17 $y = -2$. D4.18
 $x = 5$. D4.19 -3 . D4.20 $\frac{1}{3}$. D4.21 $-\frac{5}{2}$. D4.22 Perpendicular. D4.23 Parallel.
D4.24 Yes.

EXERCISES

COMPUTATION

- 4.1** Name the quadrant or axis containing each point: $(3, -2)$, $(-4, 1)$, $(0, 5)$, $(-2, -6)$.
- 4.2** Find the distance between $(1, 2)$ and $(7, 10)$.
- 4.3** Find the distance between $(-3, 4)$ and $(2, -8)$.
- 4.4** Find the midpoint of the segment joining $(-4, 7)$ and $(6, -1)$.
- 4.5** Show that the triangle with vertices $(1, 1)$, $(4, 5)$ and $(8, 2)$ is an isosceles right triangle, and find its area.
- 4.6** Find the intercepts of $3x - 4y = 12$.
- 4.7** Find the intercepts of $y = x^2 - 9$.
- 4.8** Is $(2, -1)$ on the graph of $y = 3x - 7$? Is $(-1, 4)$ on the graph of $y = x^2 + 3$?
- 4.9** Find the slope of the line through $(2, 3)$ and $(6, 11)$.
- 4.10** Find the slope of the line through $(-1, 4)$ and $(3, -2)$.
- 4.11** Find the slopes of the lines through $(5, 2)$ and $(5, -3)$, and through $(-2, 4)$ and $(3, 4)$.
- 4.12** Find the slope and y -intercept of $2x + 5y = 10$.
- 4.13** Find the line with slope 3 through $(1, -2)$.
- 4.14** Find the line through $(2, 5)$ and $(-1, -4)$.
- 4.15** Find the line through $(-3, 2)$ with slope $-\frac{2}{3}$.
- 4.16** Find the line with x -intercept 4 and y -intercept -6 .
- 4.17** Find the horizontal and the vertical line through $(3, -7)$.
- 4.18** Write $y = -\frac{3}{4}x + 2$ in standard form with integer coefficients.
- 4.19** Find the line through $(1, 4)$ parallel to $y = 2x - 5$.
- 4.20** Find the line through $(6, -1)$ perpendicular to $y = 3x + 1$.
- 4.21** Find the line through $(-2, 3)$ perpendicular to $2x - 5y = 4$.

- 4.22** Are the lines $4x - 6y = 5$ and $3x + 2y = 1$ parallel, perpendicular, or neither?
- 4.23** Show that the points $(0,0)$, $(4,1)$, $(5,5)$ and $(1,4)$ are the vertices of a parallelogram. Is it a rhombus (all sides equal)?
- 4.24** Sketch $y = -2x + 4$ and $y = \frac{1}{2}x - 1$ on the same axes. What angle do they make?
- 4.25** Graph $\frac{x}{3} + \frac{y}{5} = 1$ using its intercepts.
- 4.26** A plumber charges \$65 plus \$40 per hour. Write the cost C of an h -hour job, and find the cost of a 3.5-hour job.
- 4.27** A car is worth \$18,000 at age 3 years and \$10,000 at age 7. Assuming linear depreciation, write its value V as a function of its age t , and find when it is worth nothing.
- 4.28** Water freezes at $0^\circ\text{C} = 32^\circ\text{F}$ and boils at $100^\circ\text{C} = 212^\circ\text{F}$. Assuming a linear relationship, find F in terms of C .
- 4.29** A tree is 4 ft tall when planted and grows 1.5 ft per year. When will it be 22 ft tall?
- 4.30** Phone plan A costs \$30 per month plus \$0.05 per minute; plan B costs \$20 plus \$0.09 per minute. For how many minutes is plan A cheaper?
- 4.31** Find k so that the line through $(2, k)$ and $(5, 7)$ has slope 3.
- 4.32** Find the point on the y -axis that is equidistant from $(2, 3)$ and $(4, 1)$.

CONCEPTS AND PROOF

- 4.33** Derive the distance formula from the Pythagorean Theorem, and explain why it holds even when $x_2 < x_1$.
- 4.34** Prove that the midpoint given by Theorem 4.1 is equidistant from the two endpoints.
- 4.35** Explain why a vertical line has undefined slope and cannot be written in the form $y = mx + b$.
- 4.36** Show that the slope between any two points of the graph of $y = mx + b$ is m .

- 4.37** Prove the perpendicularity condition: the lines $y = m_1x$ and $y = m_2x$ are perpendicular if and only if $m_1m_2 = -1$. *Hint:* use the points $(1, m_1)$ and $(1, m_2)$, the origin, and the converse of the Pythagorean Theorem.
- 4.38** Show that the points $(-1, -3)$, $(1, 1)$ and $(4, 7)$ lie on one line.
- 4.39** In a cost model $C = a + bn$, what are the units of b , and what does it mean in words?

CHALLENGE

- 4.40** Find the perpendicular bisector of the segment from $(1, 2)$ to $(5, 8)$.
- 4.41** Find the area of the triangle formed by the coordinate axes and the line $3x + 4y = 24$.
- 4.42** Find the distance from the origin to the line $3x + 4y = 25$. *Hint:* the shortest path is along a perpendicular.

CHAPTER 5

FUNCTIONS

LEARNING OBJECTIVES

- Decide whether a relation is a function, and use function notation.
- Find the domain and range of a function from a formula or a graph.
- Read a graph: values, zeros, intervals of increase and decrease, and maximum and minimum values.
- Compute average rates of change, and work with linear functions.
- Recognize the graphs of the basic functions, evaluate piecewise functions, and shift graphs.

The equations of Chapter 4 relate two quantities. Often one quantity *determines* the other: the cost of a taxi ride is determined by the distance, and the area of a square by its side. A function is the mathematical form of this idea, and it is the most important idea in the rest of this book and in all of calculus.

5.1 FUNCTIONS AND FUNCTION NOTATION

Definition 5.1 (Function). A *function* f is a rule that assigns to each input x in a set D exactly one output $f(x)$. The set D is the *domain* of f , and the set of all outputs is its *range*. We write $y = f(x)$ and call x the *independent variable* and y the *dependent variable*.

The key words are *exactly one*. A rule that gives two different outputs for some input is not a function. More generally, any set of ordered pairs is called a *relation*, and a relation is a function when no two of its pairs have the same first coordinate and different second coordinates.

Example 5.2 (Functions and non-functions). (a) $\{(1, 2), (2, 3), (3, 2)\}$ is a function: each input appears once. Two inputs sharing the output 2 is allowed. (b) $\{(1, 2), (2, 3), (1, 4)\}$ is not a function: the input 1 has two outputs. (c) $y = x^2$ defines y as a function of x . But $y^2 = x$ does not: $x = 4$ gives both $y = 2$ and $y = -2$.

The notation $f(x)$ is read “ f of x ”; it means the output of f at the input x , not f times x . To evaluate f at any input, replace every x in the formula by that input, in parentheses.

Example 5.3 (Evaluating a function). Let $f(x) = x^2 - 3x + 2$. Find $f(-1)$, $f(a + 1)$, and $f(x + h) - f(x)$.

Solution. $f(-1) = (-1)^2 - 3(-1) + 2 = 6$. Next, $f(a + 1) = (a + 1)^2 - 3(a + 1) + 2 = a^2 + 2a + 1 - 3a - 3 + 2 = a^2 - a$. Finally, $f(x + h) = x^2 + 2xh + h^2 - 3x - 3h + 2$, so $f(x + h) - f(x) = 2xh + h^2 - 3h$.

COMMON ERROR ($f(a + 1) \neq f(a) + 1$). In general $f(a + 1)$, $f(a) + 1$ and $f(a) + f(1)$ are three different numbers. Here $f(a + 1) = a^2 - a$, while $f(a) + 1 = a^2 - 3a + 3$.

The graph of f is the graph of the equation $y = f(x)$. Since each x has only one y , a vertical line can meet the graph at most once.

Theorem 5.4 (Vertical Line Test). *A curve in the coordinate plane is the graph of a function of x if and only if no vertical line meets the curve more than once.*

5.2 DOMAIN AND RANGE

When a function is given by a formula with no domain stated, its domain is every real number for which the formula makes sense. For the functions of this book, two things can go wrong: a denominator can be zero, and an even root (such as a square root) can have a negative argument.

Example 5.5 (Finding domains). Find the domains of (a) $f(x) = \frac{1}{x - 4}$,
(b) $g(x) = \sqrt{2x - 6}$, (c) $h(x) = \frac{\sqrt{x + 1}}{x - 2}$.

Solution. (a) Exclude $x = 4$: the domain is $\{x \mid x \neq 4\} = (-\infty, 4) \cup (4, \infty)$.
 (b) Require $2x - 6 \geq 0$, so $x \geq 3$: the domain is $[3, \infty)$. (c) Require $x + 1 \geq 0$ and $x \neq 2$: the domain is $[-1, 2) \cup (2, \infty)$.

The range is usually harder to find from a formula. Often the graph shows it: the range is the set of y -values the graph reaches. For example, $x^2 \geq 0$ for every x , so the range of $f(x) = x^2 + 3$ is $[3, \infty)$.

5.3 READING GRAPHS

A graph shows a function's behavior at a glance. The value $f(a)$ is the height of the graph above (or below) $x = a$. The *zeros* of f , the solutions of $f(x) = 0$, are the x -intercepts.

Definition 5.6 (Increasing, decreasing, maximum, minimum). A function is *increasing* on an interval if its graph rises from left to right there: $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in the interval. It is *decreasing* if the graph falls. A *local maximum* is a value $f(c)$ that is at least as large as all nearby values (a peak), and a *local minimum* is one at least as small as all nearby values (a valley).

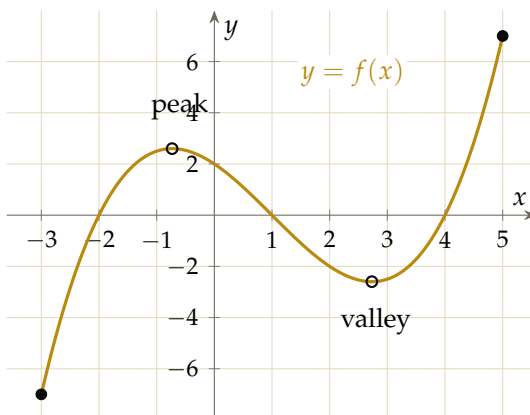


Figure 5.1: A function with domain $[-3, 5]$. Its zeros are -2 , 1 and 4 ; it has a local maximum of about 2.6 near $x = -0.7$ and a local minimum of about -2.6 near $x = 2.7$.

Example 5.7 (Reading a graph). Use Fig. 5.1 to describe f .

Solution. The domain is $[-3, 5]$, and since the lowest and highest points

are the endpoints $(-3, -7)$ and $(5, 7)$, the range is $[-7, 7]$. Reading heights: $f(0) = 2$ and $f(2) = -2$. The zeros are $x = -2, 1, 4$, and $f(x) > 0$ on $(-2, 1)$ and $(4, 5]$. The function increases on about $[-3, -0.7]$, decreases on about $[-0.7, 2.7]$, and increases again on about $[2.7, 5]$. (Locating the turning points exactly is a calculus problem; this f is in fact $\frac{1}{4}(x + 2)(x - 1)(x - 4)$.)

5.4 LINEAR FUNCTIONS AND RATES OF CHANGE

A *linear function* has the form $f(x) = mx + b$. Its graph is the line with slope m and y -intercept b , and it changes at the constant rate m . For other functions the rate of change varies, and we measure it on an interval.

Definition 5.8 (Average rate of change). The *average rate of change* of f from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a},$$

the slope of the line through $(a, f(a))$ and $(b, f(b))$.

Example 5.9 (Average rates of change). For $f(x) = x^2$, the average rate of change from 1 to 3 is $\frac{9-1}{2} = 4$, and from 3 to 5 it is $\frac{25-9}{2} = 8$. The function grows faster as x increases, so it is not linear. For a linear function, the average rate of change on every interval is the slope.

Example 5.10 (A linear function from two values). Find the linear function with $f(2) = 7$ and $f(5) = 16$.

Solution. The slope is $\frac{16-7}{5-2} = 3$, so $f(x) = 3x + b$, and $f(2) = 6 + b = 7$ gives $b = 1$. So $f(x) = 3x + 1$.

5.5 BASIC FUNCTIONS AND THEIR GRAPHS

A handful of functions appear so often that their graphs should be known by sight (Fig. 5.2).

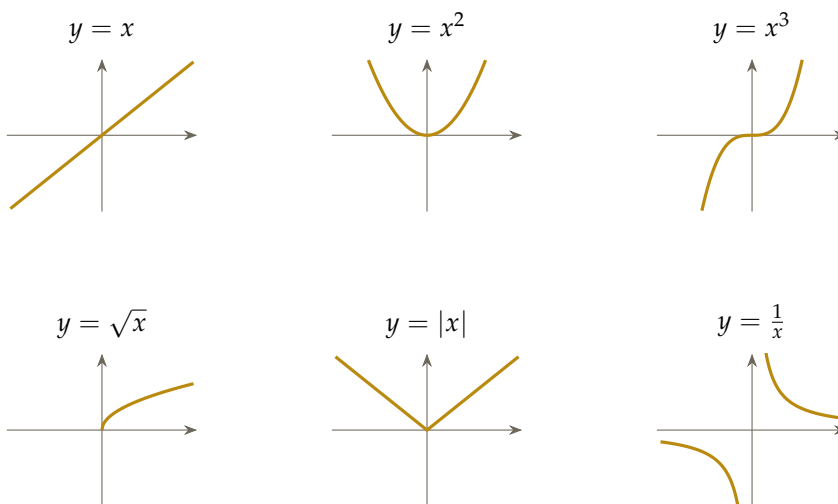


Figure 5.2: Six basic functions. Note the domains: $\sqrt{x}$ needs $x \geq 0$, and $\frac{1}{x}$ excludes 0.

5.5.1 PIECEWISE FUNCTIONS

A *piecewise* function uses different formulas on different parts of its domain. The absolute value is one: $|x| = x$ for $x \geq 0$ and $-x$ for $x < 0$.

Example 5.11 (Evaluating a piecewise function). Let $f(x) = \begin{cases} x^2, & x < 1 \\ 3x - 1, & x \geq 1. \end{cases}$ Find $f(-2)$, $f(1)$ and $f(3)$.

Solution. First decide which piece applies. $-2 < 1$, so $f(-2) = (-2)^2 = 4$. $1 \geq 1$, so $f(1) = 3(1) - 1 = 2$. And $f(3) = 8$.

5.5.2 SHIFTING GRAPHS

Adding a constant to a function moves its graph. If $k > 0$, the graph of $y = f(x) + k$ is the graph of f moved *up* k units, and the graph of $y = f(x - k)$ is the graph of f moved *right* k units. A minus sign in front, $y = -f(x)$, reflects the graph across the x -axis.

Example 5.12 (Shifting). The graph of $y = (x + 1)^2 - 4$ is the parabola $y = x^2$ moved left 1 and down 4; its lowest point moves from $(0, 0)$ to $(-1, -4)$. The graph of $y = \sqrt{x - 3} + 2$ is $y = \sqrt{x}$ moved right 3 and up 2, so it starts at $(3, 2)$; its domain is $[3, \infty)$ and its range $[2, \infty)$.

COMMON ERROR (The horizontal shift goes the “wrong” way). $y = f(x - 3)$ moves the graph to the *right*. The output f gave at x is now produced at $x + 3$, which is 3 units further right.

LOOKING AHEAD. Volume II begins by revisiting functions in more depth: composition, inverse functions, and the full set of transformations. The average rate of change $\frac{f(b)-f(a)}{b-a}$ becomes, as b approaches a , the instantaneous rate of change, the derivative of Volume III.

5.6 WORKED PROBLEMS

These problems use function notation, domains, rates of change, and piecewise definitions. State the domain whenever a function models a real situation.

PROBLEM 5.1 A domain with two conditions

Find the domain of $f(x) = \frac{\sqrt{6-2x}}{x+1}$.

Solution. The square root needs $6 - 2x \geq 0$, so $x \leq 3$. The denominator needs $x \neq -1$. Both conditions together give $(-\infty, -1) \cup (-1, 3]$.

PROBLEM 5.2 A difference quotient

For $f(x) = 3x^2 - x$, simplify $\frac{f(x+h) - f(x)}{h}$ for $h \neq 0$.

Solution. $f(x+h) = 3(x+h)^2 - (x+h) = 3x^2 + 6xh + 3h^2 - x - h$. Subtracting $f(x) = 3x^2 - x$ leaves $6xh + 3h^2 - h$. Each term has a factor of h , so dividing gives $6x + 3h - 1$. This expression is the average rate of change of f from x to $x+h$.

PROBLEM 5.3 Rates of change of temperature

The temperature t hours after 8 a.m. is $T(t) = 60 + 4t - \frac{1}{2}t^2$ degrees. Find the average rate of change from 8 a.m. to noon and from noon to 4 p.m., and interpret.

Solution. $T(0) = 60$, $T(4) = 60 + 16 - 8 = 68$ and $T(8) = 60 + 32 - 32 = 60$. From 8 to 12 the average rate is $\frac{68-60}{4} = 2$ degrees per hour; from 12 to 4 it is $\frac{60-68}{4} = -2$ degrees per hour. The morning warmed and

the afternoon cooled at the same average rate, though not necessarily at a steady rate within each period.

PROBLEM 5.4 A piecewise tax

A tax is 10% of the first \$10,000 of income and 15% of income above \$10,000. Write the tax $T(x)$ on income x , find $T(25,000)$, and find the income on which the tax is \$4000.

Solution.

$$T(x) = \begin{cases} 0.10x, & 0 \leq x \leq 10,000 \\ 1000 + 0.15(x - 10,000), & x > 10,000. \end{cases}$$

$T(25,000) = 1000 + 0.15(15,000) = 3250$. A tax of \$4000 exceeds \$1000, so the second piece applies: $1000 + 0.15(x - 10,000) = 4000$, so $x - 10,000 = 20,000$ and $x = \$30,000$.

PROBLEM 5.5 Fencing against a wall

A rectangular garden is fenced on three sides with 100 m of fence; the fourth side is a wall. Write its area as a function of the width x (the sides perpendicular to the wall), give its domain, and compare $A(20)$ and $A(25)$.

Solution. The two widths use $2x$ meters, leaving $100 - 2x$ for the side parallel to the wall. So $A(x) = x(100 - 2x)$. Both dimensions must be positive, so the domain is $0 < x < 50$. $A(20) = 20 \cdot 60 = 1200$ and $A(25) = 25 \cdot 50 = 1250$ square meters. Chapter 9 shows that $x = 25$ gives the largest possible area.

PROBLEM 5.6 Reading a shifted absolute value

For $f(x) = |x - 2| - 3$, find the zeros, the minimum value, and the range.

Solution. The graph is $y = |x|$ moved right 2 and down 3, a V with lowest point $(2, -3)$. So the minimum value is -3 and the range is $[-3, \infty)$. The zeros solve $|x - 2| = 3$: $x = 5$ or $x = -1$.

PROBLEM 5.7 Which grows faster?

Compare the average rates of change of $f(x) = 2x + 10$ and $g(x) = x^2$ on $[0, 1]$ and on $[0, 5]$.

Solution. f is linear, so its average rate is 2 on every interval. For g : on $[0, 1]$ it is $\frac{1-0}{1} = 1$, and on $[0, 5]$ it is $\frac{25-0}{5} = 5$. So g starts out growing more slowly than f but soon grows faster. Since g 's rate keeps increasing while f 's stays fixed, g eventually overtakes f in value too (at $x = 1 + \sqrt{11} \approx 4.3$, by Chapter 9).

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|--|--|
| D5.1 $f(4)$ for $f(x) = 2x + 3$ | D5.15 Average rate of change of x^3 on $[1, 2]$ |
| D5.2 $f(-1)$ for $f(x) = x^2 - x$ | D5.16 Linear f with slope 2 and $f(0) = 5$ |
| D5.3 $g(0)$ for $g(x) = 5 - x^2$ | D5.17 Linear f with $f(1) = 4$ and $f(3) = 10$ |
| D5.4 $h(2)$ for $h(x) = \frac{1}{x+2}$ | D5.18 Zero of $f(x) = 4x - 12$ |
| D5.5 Is $\{(1, 2), (3, 2)\}$ a function? | D5.19 $f(-6)$ for $f(x) = x $ |
| D5.6 Is $\{(0, 1), (0, -1)\}$ a function? | D5.20 $f(2)$ for $f(x) = \begin{cases} 1, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$ |
| D5.7 Domain of $\frac{1}{x-7}$ | D5.21 $f(-5)$ for the function in D5.20 |
| D5.8 Domain of $\sqrt{x+5}$ | D5.22 How is $y = x^2 + 3$ obtained from $y = x^2$? |
| D5.9 Domain of $x^2 + 1$ | D5.23 How is $y = x - 4 $ obtained from $y = x $? |
| D5.10 Domain of $\frac{\sqrt{x}}{x-1}$ | D5.24 Domain of $y = \sqrt{x+2}$ |
| D5.11 Range of x^2 | |
| D5.12 Range of $\sqrt{x} + 1$ | |
| D5.13 Average rate of change of x^2 on $[0, 2]$ | |
| D5.14 Average rate of change of $3x - 1$ on $[-5, 5]$ | |

ANSWERS TO SKILL DRILLS

D5.1 11. D5.2 2. D5.3 5. D5.4 $\frac{1}{4}$. D5.5 Yes. D5.6 No. D5.7 $x \neq 7$. D5.8 $x \geq -5$. D5.9 All real numbers. D5.10 $x \geq 0, x \neq 1$. D5.11 $[0, \infty)$. D5.12 $[1, \infty)$. D5.13 2. D5.14 3. D5.15 7. D5.16 $f(x) = 2x + 5$. D5.17 $f(x) = 3x + 1$. D5.18 $x = 3$. D5.19 6. D5.20 3. D5.21 1. D5.22 Shift up 3. D5.23 Shift right 4. D5.24 $[-2, \infty)$.

EXERCISES

COMPUTATION

- 5.1** Which of these define y as a function of x ? (a) $\{(1, 3), (2, 5), (3, 3)\}$; (b) $\{(2, 1), (2, 4), (3, 5)\}$; (c) $y = 2x - 1$; (d) $x = y^2$.
- 5.2** Which of these curves pass the vertical line test: a circle; the parabola $y = x^2$; a horizontal line; a vertical line?
- 5.3** For $f(x) = 3x - 5$, find $f(0)$, $f(-2)$ and $f(\frac{1}{3})$.
- 5.4** For $g(x) = x^2 - 2x$, find $g(3)$, $g(-1)$, $g(a)$ and $g(a + 2)$.
- 5.5** For $h(t) = \frac{t + 1}{t - 2}$, find $h(0)$, $h(3)$ and $h(-1)$.
- 5.6** For $f(x) = 2x^2 + 1$, simplify $f(x + h) - f(x)$.
- 5.7** Find the domain of $f(x) = 5x - 3$.
- 5.8** Find the domain of $g(x) = \frac{1}{x + 6}$.
- 5.9** Find the domain of $h(x) = \sqrt{10 - 2x}$.
- 5.10** Find the domain of $k(x) = \frac{x}{x^2 - 9}$.
- 5.11** Find the domain of $p(x) = \frac{\sqrt{x - 1}}{x - 5}$.
- 5.12** Find the range of $f(x) = x^2 + 3$ and of $g(x) = -|x|$.
- 5.13** For the function in Fig. 5.1, find $f(-1)$ and $f(3)$, and the intervals on which $f(x) < 0$.
- 5.14** Find the average rate of change of $f(x) = x^2 - 4x$ from $x = 1$ to $x = 5$.
- 5.15** Find the average rate of change of $f(x) = \sqrt{x}$ from $x = 4$ to $x = 9$.

- 5.16** A town had 45,000 people in 2010 and 52,500 in 2020. Find the average rate of change of the population, with units.
- 5.17** Find the linear function with $f(2) = 7$ and $f(5) = 16$.
- 5.18** Find the linear function with $f(0) = -4$ and slope $\frac{1}{2}$, and find $f(10)$.
- 5.19** Is $f(x) = 4 - 3x$ increasing or decreasing? At what rate does it change?
- 5.20** For $f(x) = \begin{cases} x^2, & x < 1 \\ 3x - 1, & x \geq 1 \end{cases}$, find $f(-2)$, $f(1)$ and $f(3)$, and sketch the graph.
- 5.21** A courier charges \$6 for packages up to 2 lb, plus \$1.50 for each pound over 2 lb. Write the cost $C(w)$ as a piecewise function and find $C(1)$, $C(5)$ and $C(10)$.
- 5.22** Write $f(x) = |2x - 4|$ as a piecewise function without absolute value signs.
- 5.23** Describe how to obtain the graph of $y = \sqrt{x - 3} + 2$ from $y = \sqrt{x}$, and give its domain and range.
- 5.24** Describe how to obtain the graph of $y = (x + 1)^2 - 4$ from $y = x^2$, and find its lowest point.
- 5.25** Describe how to obtain the graphs of $y = |x| - 5$ and $y = -x^2$ from basic graphs.
- 5.26** Let $f(x) = x^2 - 1$ and $g(x) = 2x + 3$. Find $f(2) + g(2)$, $f(g(1))$ and $g(f(1))$.
- 5.27** Let $f(x) = \frac{1}{x}$. Find $f(f(x))$ for $x \neq 0$.
- 5.28** A rectangle has perimeter 20 m. Write its area A as a function of its width w , and give the domain that makes sense.
- 5.29** A taxi charges \$3.50 plus \$2.25 per mile. Write the cost $C(m)$ of an m -mile ride, find $C(8)$, and find how far one can ride for \$30.
- 5.30** An open box is made from a 12×12 inch sheet by cutting $x \times x$ squares from the corners and folding up the sides. Write its volume $V(x)$, give a sensible domain, and find $V(2)$.
- 5.31** For $f(x) = 2x + 1$, solve $f(x) = 9$ and $f(x) > 0$.
- 5.32** For $g(x) = x^2$, solve $g(x) = 16$.

CONCEPTS AND PROOF

- 5.33** Explain why the vertical line test identifies the graphs of functions.
- 5.34** Can the graph of a function have two different y -intercepts? Two different x -intercepts? Explain.
- 5.35** Show that the average rate of change of $f(x) = mx + b$ on every interval $[a, b]$ equals m .
- 5.36** Show by example that $f(a + b) = f(a) + f(b)$ is false for $f(x) = x^2$. For which linear functions $f(x) = mx + b$ is it true for all a, b ?
- 5.37** Explain why $\sqrt{x}$ is undefined for $x < 0$, while $\sqrt[3]{x}$ is defined for every real x .
- 5.38** Give a formula for a function whose domain is all real numbers except 2 and 5.
- 5.39** Describe the graph of $f(x) = \frac{|x|}{x}$, and give its domain and range.

CHALLENGE

- 5.40** Find all linear functions f with $f(f(x)) = 9x - 8$.
- 5.41** Let $f(x) = \frac{x+1}{x-1}$. Show that $f(f(x)) = x$ for $x \neq 1$.
- 5.42** A function satisfies $f(x + 1) = f(x) + 3$ for every x , and $f(0) = 2$. Find $f(10)$, and find $f(n)$ for every positive integer n .

CHAPTER 6

SYSTEMS OF LINEAR EQUATIONS

LEARNING OBJECTIVES

- Solve systems of two linear equations by graphing, substitution, and elimination.
- Recognize systems with no solution or infinitely many solutions.
- Model problems with two or three unknowns as systems and solve them.
- Solve systems of three linear equations in three variables by elimination.
- Graph linear inequalities in two variables and systems of such inequalities.

Many problems involve two or more unknown quantities linked by two or more conditions. Each condition is an equation, and a solution must satisfy all of them at once. This chapter develops methods for such *systems* of linear equations, and ends with systems of inequalities, which describe regions of the plane.

6.1 SYSTEMS AND THEIR GRAPHS

Definition 6.1 (System of equations). A *system of equations* is a collection of equations in the same variables. A *solution* of the system is an assignment of values to the variables that satisfies every equation. A system with at least one solution is *consistent*; one with no solution is *inconsistent*.

A solution of two linear equations in x and y is a point on both lines, so it is an intersection point. Two lines in the plane meet in exactly one point, or are parallel and never meet, or are the same line. This gives the three possible cases.

Theorem 6.2 (Number of solutions). A system of two linear equations in two variables has exactly one solution (the lines intersect), no solution (the lines are parallel and distinct), or infinitely many solutions (the equations describe the same line).

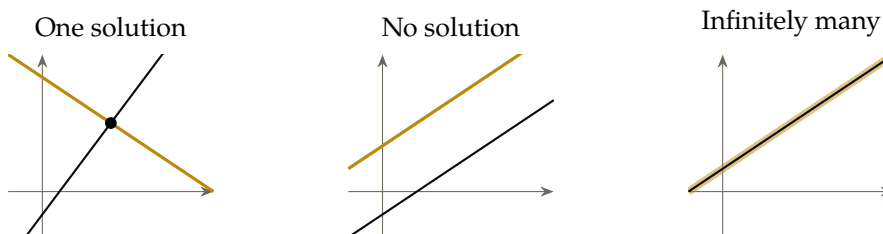


Figure 6.1: Two lines meet once, never, or everywhere.

Example 6.3 (Solving by graphing). Solve $y = 2x - 1$, $y = -x + 5$ by graphing.

Solution. The lines cross at $(2, 3)$ (Fig. 6.1, left). Check in both equations: $2(2) - 1 = 3$ and $-2 + 5 = 3$. Graphing shows the situation clearly but gives exact answers only when the intersection happens to be at a readable point; the algebraic methods below are exact.

6.2 SUBSTITUTION

Substitution solves one equation for one variable and substitutes the result into the other equation, which then has only one variable. It is most convenient when some variable has coefficient 1 or -1 .

Example 6.4 (Substitution). Solve $x - 2y = 1$, $3x - 4y = 7$.

Solution. The first equation gives $x = 1 + 2y$. Substituting into the second: $3(1 + 2y) - 4y = 7$, so $3 + 2y = 7$ and $y = 2$. Then $x = 1 + 2(2) = 5$. The solution is $(5, 2)$. Check: $5 - 4 = 1$ and $15 - 8 = 7$.

6.3 ELIMINATION

Elimination adds multiples of the equations so that one variable cancels. It rests on a simple fact: if $A = B$ and $C = D$, then $A + C = B + D$, and multiplying an equation by a nonzero constant does not change its solutions.

Example 6.5 (Elimination). Solve $4x + 3y = 10$, $6x - 5y = -4$.

Solution. To eliminate x , multiply the first equation by 3 and the second by 2, making both x -coefficients 12:

$$12x + 9y = 30, \quad 12x - 10y = -8.$$

Subtracting the second from the first: $19y = 38$, so $y = 2$. Then $4x + 6 = 10$ gives $x = 1$. The solution is $(1, 2)$.

When elimination removes *both* variables, the system is one of the special cases.

Example 6.6 (The special cases). (a) $2x - 4y = 6$ and $x - 2y = 3$: twice the second equation is the first, so both describe the same line. Every point of the line $x - 2y = 3$ is a solution; we can write them as $(x, \frac{x-3}{2})$ for any x . (b) $x - 2y = 3$ and $x - 2y = 5$: subtracting gives $0 = -2$, a contradiction. The lines are parallel, and there is no solution.

COMMON ERROR (A false statement is information). When the variables disappear and leave a false statement such as $0 = -2$, the conclusion is “no solution,” not that an arithmetic error was made. When they leave a true statement such as $0 = 0$, the conclusion is “infinitely many solutions,” not “ $x = 0$.”

6.3.1 THREE EQUATIONS IN THREE VARIABLES

The same idea solves larger systems: use one equation to eliminate a variable from the others, reducing three equations in three unknowns to two equations in two unknowns.

Example 6.7 (A 3×3 system). Solve $x + y + z = 6$, $x - y + 2z = 5$, $2x + y - z = 1$.

Solution. Add the first two equations to eliminate y : $2x + 3z = 11$. Add the second and third: $3x + z = 6$. From the last, $z = 6 - 3x$; substituting, $2x + 18 - 9x = 11$, so $x = 1$ and $z = 3$. Then $y = 6 - 1 - 3 = 2$. The solution is $(1, 2, 3)$, and it satisfies all three original equations.

6.4 APPLICATIONS

With two unknowns, a word problem needs two independent conditions. Assign a variable to each unknown, write one equation per condition, and solve the system.

Example 6.8 (Ticket sales). A theater sold 220 tickets for \$1500. Adult tickets cost \$9 and child tickets \$5. How many of each were sold?

Solution. With a adult and c child tickets: $a + c = 220$ (number) and $9a + 5c = 1500$ (money). Substituting $c = 220 - a$: $9a + 1100 - 5a = 1500$, so $a = 100$ and $c = 120$.

Example 6.9 (Current). A boat travels 36 miles downstream in 2 hours and returns upstream in 3 hours. Find the speed of the boat in still water and of the current.

Solution. With boat speed b and current c , the downstream speed is $b + c = \frac{36}{2} = 18$ and the upstream speed is $b - c = \frac{36}{3} = 12$. Adding, $2b = 30$, so $b = 15$ mph and $c = 3$ mph.

Example 6.10 (A blend). Coffee selling for \$8 per pound is mixed with coffee selling for \$12 per pound to make 20 pounds of a blend worth \$9.50 per pound. How much of each is used?

Solution. $x + y = 20$ (pounds) and $8x + 12y = 9.50(20) = 190$ (value). Substituting $y = 20 - x$: $8x + 240 - 12x = 190$, so $x = 12.5$ and $y = 7.5$ pounds.

6.5 LINEAR INEQUALITIES IN TWO VARIABLES

The graph of a linear inequality such as $y < 2x + 1$ is a *half-plane*. Graph the boundary line $y = 2x + 1$, dashed if the inequality is strict ($<$ or $>$) and solid otherwise ($\leq$ or $\geq$). Then shade the side that satisfies the inequality, found by testing one point not on the line, often the origin. Here $0 < 2(0) + 1$ is true, so the side containing the origin, below the line, is shaded.

The solution of a *system* of inequalities is the set of points satisfying all of them: the region where the shaded half-planes overlap. Its corners are found by solving pairs of boundary equations.

Example 6.11 (A system of inequalities). Graph $x + y \leq 6$, $y \geq x$, $x \geq 0$, and find the corners of the region.

Solution. Testing $(1, 2)$, which satisfies all three, shows the region lies below $x + y = 6$, above $y = x$, and right of the y -axis: the triangle in Fig. 6.2. Its corners are $(0, 0)$, $(0, 6)$, and the intersection of $x + y = 6$ with $y = x$, which is $(3, 3)$.

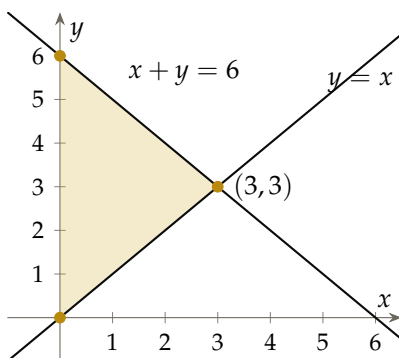


Figure 6.2: The region $x + y \leq 6$, $y \geq x$, $x \geq 0$ is a triangle.

COMMON ERROR (Choose a test point off the line). The origin is a convenient test point only when it is not on the boundary. For $y > 2x$, the origin lies on the line $y = 2x$; test $(0, 1)$ instead.

LOOKING AHEAD. Volume II solves larger systems with matrices and Gaussian elimination, and uses systems of inequalities for *linear programming*, finding the best corner of a region like Fig. 6.2. Systems involving curves, such as a line and a parabola, appear in Chapter 9.

6.6 WORKED PROBLEMS

These problems use substitution and elimination in context. Check each solution in every original equation.

PROBLEM 6.1 A system with fractions

Solve $\frac{x+1}{3} + \frac{y}{2} = 3$, $\frac{x}{4} - \frac{y-2}{3} = \frac{5}{4}$.

Solution. Clear the fractions first. Multiplying the first equation by 6: $2x + 2 + 3y = 18$, so $2x + 3y = 16$. Multiplying the second by 12: $3x - 4y + 8 = 15$, so $3x - 4y = 7$. Eliminate y : four times the first plus three times the second gives $17x = 85$, so $x = 5$, and then $3y = 6$, $y = 2$. Check: $\frac{6}{3} + 1 = 3$ and $\frac{5}{4} - 0 = \frac{5}{4}$.

PROBLEM 6.2 Supply and demand

The demand for a product is $p = 80 - 2q$ and the supply is $p = 20 + 4q$, where p is the price in dollars and q the quantity in thousands. Find the equilibrium point, where supply equals demand.

Solution. Set the prices equal: $80 - 2q = 20 + 4q$, so $6q = 60$ and $q = 10$. Then $p = 80 - 20 = 60$. At \$60, buyers want 10,000 units and sellers supply 10,000 units.

PROBLEM 6.3 A rectangle

A rectangle has perimeter 46 cm, and its length is 1 cm less than three times its width. Find its dimensions.

Solution. $2l + 2w = 46$ and $l = 3w - 1$. Substituting: $2(3w - 1) + 2w = 46$, so $8w = 48$, $w = 6$ and $l = 17$. Check: $2(17) + 2(6) = 46$, and $17 = 3(6) - 1$.

PROBLEM 6.4 Three kinds of coins

A jar has 30 coins (pennies, nickels and dimes) worth \$1.35. There are three more dimes than nickels. How many of each are there?

Solution. With p, n, d coins of each kind: $p + n + d = 30$, $p + 5n + 10d = 135$ (in cents), and $d = n + 3$. Subtracting the first equation from the second eliminates p : $4n + 9d = 105$. Substituting $d = n + 3$: $13n + 27 = 105$, so $n = 6$, $d = 9$ and $p = 15$. Check: $15 + 30 + 90 = 135$ cents.

PROBLEM 6.5 A system with a parameter

For which values of a does the system $ax + y = 2$, $4x + ay = 4$ have exactly one solution, no solution, or infinitely many?

Solution. The slopes are $-a$ and $-\frac{4}{a}$ (for $a \neq 0$). They are equal when $a^2 = 4$. So for $a \neq \pm 2$ the lines have different slopes and meet once. (For $a = 0$ the lines $y = 2$ and $x = 1$ also meet once.) If $a = 2$: the equations

are $2x + y = 2$ and $4x + 2y = 4$, the same line, so there are infinitely many solutions. If $a = -2$: they are $-2x + y = 2$ and $4x - 2y = 4$; multiplying the first by -2 gives $4x - 2y = -4$, which contradicts the second, so there is no solution.

PROBLEM 6.6 A feasible region

Find the corners of the region $x \geq 0, y \geq 0, 2x + y \leq 10, x + 2y \leq 8$.

Solution. The region is bounded by the axes and the two lines. The corners on the axes are $(0, 0)$, $(5, 0)$ (where $2x + y = 10$ meets $y = 0$), and $(0, 4)$ (where $x + 2y = 8$ meets $x = 0$). The fourth corner is where the two lines meet: subtracting twice the second equation from the first gives $-3y = -6$, so $y = 2$ and $x = 4$. Check that each corner satisfies all four inequalities; for example $(5, 0)$ gives $5 \leq 8$.

PROBLEM 6.7 Testing three points

Do the points $(1, 2)$, $(3, 8)$ and $(-2, -7)$ lie on one line?

Solution. Find the line $y = ax + b$ through the first two points: $a + b = 2$ and $3a + b = 8$. Subtracting, $2a = 6$, so $a = 3$ and $b = -1$. The line is $y = 3x - 1$. At $x = -2$ it gives $y = -7$, so the third point is on it too. The three points are collinear.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D6.1 Is $(1, 2)$ a solution of $x + y = 3$,
 $x - y = -1$?

D6.2 $x + y = 5, x - y = 1$

D6.3 $y = 2x, x + y = 9$

D6.4 $2x + y = 7, x = 2$

D6.5 $x + 2y = 8, x - 2y = 0$

D6.6 $3x + y = 10, x + y = 4$

D6.7 $y = x + 1, y = 3x - 3$

D6.8 $2x + 3y = 13, x = y + 4$

D6.9 $x - y = 2, 2x - 2y = 4$

D6.10 $x + y = 1, x + y = 2$

D6.11 $5x + 2y = 1, 3x - 2y = 7$

D6.12 $4x - y = 9, 2x + y = 3$

D6.13 $x + y + z = 3, y + z = 2, z = 1$

D6.14 How many solutions if the
lines have different slopes?

- D6.15** How many if the slopes are equal and the intercepts differ?
for $y \geq -x + 1$?
- D6.16** Two numbers with sum 20 and difference 6
- D6.17** Is $(0, 0)$ a solution of $y > x - 2$?
- D6.18** Is $(3, 0)$ a solution of $2x + y \leq 5$?
- D6.19** Is the boundary of $y < 4$ solid or dashed?
- D6.20** Shade above or below the line
- D6.21** Corner where $x = 1$ meets $y = 2$
- D6.22** Intersection of $y = x$ and $y = -x + 4$
- D6.23** k for which $y = kx + 1$ is parallel to $y = 3x$
- D6.24** Downstream speed 10 mph, upstream 6 mph: speed in still water

ANSWERS TO SKILL DRILLS

D6.1 Yes. D6.2 $(3, 2)$. D6.3 $(3, 6)$. D6.4 $(2, 3)$. D6.5 $(4, 2)$. D6.6 $(3, 1)$. D6.7 $(2, 3)$. D6.8 $(5, 1)$. D6.9 Infinitely many. D6.10 No solution. D6.11 $(1, -2)$. D6.12 $(2, -1)$. D6.13 $(1, 1, 1)$. D6.14 One. D6.15 None. D6.16 13 and 7. D6.17 Yes. D6.18 No. D6.19 Dashed. D6.20 Above. D6.21 $(1, 2)$. D6.22 $(2, 2)$. D6.23 3. D6.24 8 mph.

EXERCISES

COMPUTATION

- 6.1** Is $(3, -1)$ a solution of the system $2x + y = 5$, $x - 3y = 6$?
- 6.2** Solve by graphing: $y = x + 1$, $y = -2x + 7$.
- 6.3** Solve by substitution: $y = 3x - 2$, $2x + y = 8$.
- 6.4** Solve by substitution: $x + 4y = 9$, $3x - 2y = 13$.
- 6.5** Solve by substitution: $2x - y = 7$, $5x + 3y = 1$.
- 6.6** Solve by elimination: $x + y = 10$, $x - y = 4$.
- 6.7** Solve by elimination: $3x + 2y = 7$, $5x - 2y = 1$.
- 6.8** Solve: $2x + 3y = 12$, $5x - 4y = 7$.
- 6.9** Solve: $3x - 5y = -13$, $4x + 3y = 2$.
- 6.10** Solve: $\frac{x}{2} + \frac{y}{3} = 4$, $\frac{x}{4} - \frac{y}{6} = 0$.

- 6.11** Solve: $0.2x + 0.5y = 1.9$, $0.3x - 0.1y = 0.3$.
- 6.12** Solve: $6x - 2y = 4$, $y = 3x - 2$.
- 6.13** Solve: $4x + 6y = 9$, $2x + 3y = 4$.
- 6.14** Find k so that the system $kx + 2y = 5$, $3x + y = 1$ has no solution.
- 6.15** Solve: $x + y + z = 6$, $x - y + 2z = 5$, $2x + y - z = 1$.
- 6.16** Solve: $x + 2y - z = 1$, $2x - y + z = 6$, $x + y + 2z = 9$.
- 6.17** Two numbers have sum 50 and difference 14. Find them.
- 6.18** A school sold 300 tickets for \$2850. Adult tickets cost \$12 and student tickets \$7. How many of each were sold?
- 6.19** A boat travels 48 miles downstream in 3 hours and returns in 4 hours. Find the boat's speed in still water and the speed of the current.
- 6.20** A plane flies 1200 miles with the wind in 4 hours and returns against the wind in 5 hours. Find the plane's airspeed and the wind speed.
- 6.21** How many pounds of \$10 coffee and \$16 coffee make 30 pounds of a blend worth \$12 per pound?
- 6.22** \$12,000 is invested in two accounts paying 5% and 8%, earning \$780 in a year. How much is in each account?
- 6.23** A product costs \$2000 to set up plus \$15 per unit, and sells for \$40 per unit. How many units must be sold to break even?
- 6.24** Car rental A charges \$40 plus \$0.20 per mile; B charges \$25 plus \$0.35 per mile. For what mileage do they cost the same, and what is that cost?
- 6.25** Find a and b so that the line $y = ax + b$ passes through $(1, 5)$ and $(3, 11)$.
- 6.26** The digits of a two-digit number add to 11. Reversing the digits increases the number by 27. Find the number.
- 6.27** Graph $y \geq 2x - 3$.
- 6.28** Graph $3x - 2y < 6$.
- 6.29** Graph the system $x + y \leq 6$, $y \geq x$, $x \geq 0$, and find its corners.
- 6.30** Graph the system $y > -1$, $x < 2$, $y \leq x + 1$, and find the corners of the region.

CONCEPTS AND PROOF

- 6.31** Explain, using graphs, why a system of two linear equations in two variables cannot have exactly two solutions.
- 6.32** Show that if a system of two linear equations has two different solutions, it has infinitely many.
- 6.33** Explain why the elimination method produces an equation with the same solutions as the system.
- 6.34** Write a system of two linear equations whose only solution is $(2, -5)$.
- 6.35** For which values of c does the system $x + 2y = 4$, $2x + 4y = c$ have infinitely many solutions? No solution?
- 6.36** Why can't the origin be used as a test point for $y > 2x$? What point could be used instead?

CHALLENGE

- 6.37** Solve $\frac{1}{x} + \frac{1}{y} = 5$, $\frac{2}{x} - \frac{3}{y} = -5$. *Hint: let $u = \frac{1}{x}$ and $v = \frac{1}{y}$.*
- 6.38** A chemist needs 100 L of a 32% acid solution, mixed from 10%, 30% and 50% solutions, using twice as much of the 50% solution as of the 10% solution. How much of each is needed?
- 6.39** Find the intersection of the lines $y = mx + 1$ and $y = 2x + m$ in terms of m , for $m \neq 2$.



PART III

POLYNOMIALS AND QUADRATICS

Operations on polynomials, factoring,
and quadratic equations and functions.

CHAPTER 7

POLYNOMIALS

LEARNING OBJECTIVES

- Identify the degree, terms, and leading coefficient of a polynomial, and write it in standard form.
- Add, subtract, and multiply polynomials.
- Recognize and use the special products: squares of binomials, the difference of squares, and cubes of binomials.
- Divide polynomials by monomials and by binomials, using long division and synthetic division.

Polynomials are the expressions built from a variable using only addition, subtraction, and multiplication. They are the simplest functions after the linear ones, and they can approximate almost any other function well. This chapter develops their arithmetic, which mirrors the arithmetic of whole numbers closely: polynomials add, multiply, and divide with remainders much as integers do.

7.1 POLYNOMIALS AND THEIR SUMS

Definition 7.1 (Polynomial). A *monomial* in x is a term ax^k , where a is a real number and k is a whole number. A *polynomial* is a sum of monomials:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If $a_n \neq 0$, the *degree* of P is n and a_n is the *leading coefficient*. Written with exponents decreasing, as here, the polynomial is in *standard form*. Polynomials with one, two and three terms are called monomials, binomials and trinomials.

Expressions such as $\frac{1}{x} = x^{-1}$ and $\sqrt{x}$ are not polynomials, because their expo-

nents are not whole numbers. A nonzero constant is a polynomial of degree 0.

Example 7.2 (Standard form and degree). $5x^3 - 2x^7 + x - 4$ in standard form is $-2x^7 + 5x^3 + x - 4$. Its degree is 7 and its leading coefficient is -2 .

Polynomials are added and subtracted by combining like terms. When subtracting, distribute the minus sign over every term of the polynomial being subtracted.

Example 7.3 (Sums and differences). (a) $(3x^2 - 5x + 2) + (x^3 - 2x^2 + 7) = x^3 + x^2 - 5x + 9$.
(b) $(4y^2 - 3y + 1) - (y^2 + 2y - 6) = 4y^2 - 3y + 1 - y^2 - 2y + 6 = 3y^2 - 5y + 7$.

7.2 PRODUCTS OF POLYNOMIALS

To multiply two polynomials, use the distributive property repeatedly: multiply every term of the first by every term of the second, then combine like terms. For two binomials, the four products are often remembered as *FOIL*: First, Outer, Inner, Last.

Example 7.4 (Multiplying). (a) $(2x + 3)(x - 5) = 2x^2 - 10x + 3x - 15 = 2x^2 - 7x - 15$.
(b) $(x - 2)(x^2 + 3x - 4) = x^3 + 3x^2 - 4x - 2x^2 - 6x + 8 = x^3 + x^2 - 10x + 8$.

In (b), each of the 2 terms of the first factor multiplies each of the 3 terms of the second, giving $2 \cdot 3 = 6$ products before combining. Counting the products is a useful check that none were missed. The degree of a product is the sum of the degrees: here $1 + 2 = 3$.

7.3 SPECIAL PRODUCTS

Three products occur so often that they should be recognized on sight. Each follows from multiplying out.

Theorem 7.5 (Special products).

$$\begin{aligned}(a + b)^2 &= a^2 + 2ab + b^2, & (a - b)^2 &= a^2 - 2ab + b^2, \\(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, & (a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3, \\(a + b)(a - b) &= a^2 - b^2.\end{aligned}$$

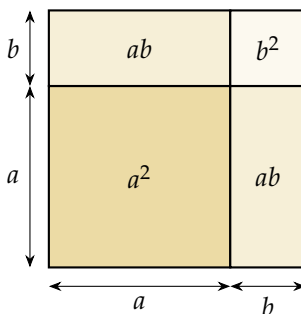


Figure 7.1: A square of side $a + b$ has area $a^2 + 2ab + b^2$. The two rectangles of area ab are the middle term that is so often forgotten.

Example 7.6 (Using the special products). (a) $(3x + 4)^2 = 9x^2 + 2(3x)(4) + 16 = 9x^2 + 24x + 16$. (b) $(2a - 5b)^2 = 4a^2 - 20ab + 25b^2$. (c) $(x^3 + 2)(x^3 - 2) = x^6 - 4$. (d) $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$.

COMMON ERROR (The middle term). $(x + 5)^2 \neq x^2 + 25$. The square of a sum always has the middle term $2ab$; here $(x + 5)^2 = x^2 + 10x + 25$. Check with $x = 1$: $36 \neq 26$.

The difference of squares gives a neat mental-arithmetic trick: $47 \cdot 53 = (50 - 3)(50 + 3) = 2500 - 9 = 2491$.

7.4 DIVIDING POLYNOMIALS

To divide a polynomial by a monomial, divide each term: $\frac{12x^4 - 8x^3 + 2x}{4x} = 3x^3 - 2x^2 + \frac{1}{2}$.

To divide by a polynomial with more than one term, use *long division*, which works exactly as for whole numbers: divide the leading terms, multiply, subtract,

bring down, and repeat until the remainder has smaller degree than the divisor.

Example 7.7 (Long division). Divide $2x^3 - 3x^2 + 4x - 5$ by $x - 2$.

Solution.

$$\begin{array}{r}
 2x^2 + x + 6 \\
 x - 2 \overline{) 2x^3 - 3x^2 + 4x - 5} \\
 \underline{- 2x^3 + 4x^2} \\
 x^2 + 4x \\
 \underline{- x^2 + 2x} \\
 6x - 5 \\
 \underline{- 6x + 12} \\
 7
 \end{array}$$

At each step, divide the leading term of what remains by x : first $\frac{2x^3}{x} = 2x^2$, then $\frac{x^2}{x} = x$, then $\frac{6x}{x} = 6$. The quotient is $2x^2 + x + 6$ and the remainder is 7, so

$$\frac{2x^3 - 3x^2 + 4x - 5}{x - 2} = 2x^2 + x + 6 + \frac{7}{x - 2}.$$

Theorem 7.8 (Division Algorithm). If $P(x)$ and $D(x)$ are polynomials and $D(x) \neq 0$, there are unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x)Q(x) + R(x),$$

where $R(x)$ is 0 or has degree less than the degree of $D(x)$. Q is the quotient and R the remainder.

The equation $P = DQ + R$ is also the check: $(x - 2)(2x^2 + x + 6) + 7 = 2x^3 - 3x^2 + 4x - 12 + 7 = 2x^3 - 3x^2 + 4x - 5$.

COMMON ERROR (Missing terms). When a power of x is missing from the dividend, write it with coefficient 0 before dividing. To divide $x^3 - 8$ by $x - 2$, use $x^3 + 0x^2 + 0x - 8$; otherwise the columns do not line up. The quotient is $x^2 + 2x + 4$ with remainder 0.

7.4.1 SYNTHETIC DIVISION

For a divisor of the form $x - c$, *synthetic division* is a shortcut that keeps only the coefficients. Write c and the coefficients of the dividend. Bring down the first

coefficient; then repeatedly multiply the latest number by c , write the product under the next coefficient, and add.

Example 7.9 (Synthetic division). Divide $2x^3 - 3x^2 + 4x - 5$ by $x - 2$ synthetically.

Solution.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & 7 \end{array}$$

The last number, 7, is the remainder, and the others, 2, 1, 6, are the coefficients of the quotient $2x^2 + x + 6$, as before.

LOOKING AHEAD. Factoring, the reverse of multiplying, is the subject of Chapter 8. In Volume II, polynomial division leads to the Remainder and Factor Theorems: the remainder on division by $x - c$ is exactly $P(c)$, as the example above illustrates ($P(2) = 16 - 12 + 8 - 5 = 7$). In Volume III, polynomials are the functions whose derivatives and integrals are easiest to compute.

7.5 WORKED PROBLEMS

These problems practice polynomial arithmetic. Look for special products before multiplying term by term; they save time and reduce errors.

PROBLEM 7.1 A difference of two squares of binomials

Simplify $(x + 3)^2 - (x - 3)^2$.

Solution. Expanding: $(x^2 + 6x + 9) - (x^2 - 6x + 9) = 12x$. Alternatively, treat it as $A^2 - B^2 = (A + B)(A - B)$ with $A = x + 3$, $B = x - 3$: $(2x)(6) = 12x$. The second method avoids expanding at all.

PROBLEM 7.2 Three factors

Multiply $(x - 1)(x + 2)(x - 3)$.

Solution. Multiply two factors first: $(x - 1)(x + 2) = x^2 + x - 2$. Then $(x^2 + x - 2)(x - 3) = x^3 - 3x^2 + x^2 - 3x - 2x + 6 = x^3 - 2x^2 - 5x + 6$. Check at $x = 1$: the original is 0, and $1 - 2 - 5 + 6 = 0$.

PROBLEM 7.3 A hidden difference of squares

Multiply $(a + b + c)(a + b - c)$.

Solution. Group $a + b$ as a single quantity: $[(a + b) + c][(a + b) - c] = (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2$. Multiplying the nine products out directly gives the same result with more work.

PROBLEM 7.4 Dividing by a quadratic

Divide $2x^4 - 3x^2 + x - 4$ by $x^2 + x - 1$.

Solution. Insert $0x^3$ for the missing term and divide:

$$\begin{array}{r}
 \\
 x^2 + x - 1 \overline{) 2x^4 + x - 4} \\
 \underline{- 2x^4 - 2x^3 + 2x^2} \\
 - 2x^3 - x^2 + x \\
 \underline{2x^3 + 2x^2 - 2x} \\
 x^2 - x - 4 \\
 \underline{- x^2 - x + 1} \\
 - 2x - 3
 \end{array}$$

The quotient is $2x^2 - 2x + 1$ and the remainder is $-2x - 3$, which has degree 1, less than the degree 2 of the divisor. So $2x^4 - 3x^2 + x - 4 = (x^2 + x - 1)(2x^2 - 2x + 1) - 2x - 3$.

PROBLEM 7.5 An open box

Squares of side x are cut from the corners of a 20×30 cm sheet, and the sides are folded up to form an open box. Write the volume as a polynomial in standard form, and find it for $x = 5$.

Solution. The base is $(20 - 2x)$ by $(30 - 2x)$ and the height is x , so $V = x(20 - 2x)(30 - 2x) = x(600 - 100x + 4x^2) = 4x^3 - 100x^2 + 600x$. At $x = 5$: $V = 5 \cdot 10 \cdot 20 = 1000 \text{ cm}^3$; the polynomial gives $500 - 2500 + 3000 = 1000$ as well.

PROBLEM 7.6 An unknown coefficient

Find a so that $(x + a)(x - 4) = x^2 + 3x - 28$.

Solution. Expanding, $(x + a)(x - 4) = x^2 + (a - 4)x - 4a$. Matching the x -coefficients: $a - 4 = 3$, so $a = 7$. The constant term confirms it: $-4(7) = -28$.

PROBLEM 7.7 Evaluating with synthetic division

Use synthetic division to find the remainder when $P(x) = x^4 - 3x^3 + 2x - 7$ is divided by $x - 3$, and compare it with $P(3)$.

Solution.

$$\begin{array}{r|rrrrrr} 3 & 1 & -3 & 0 & 2 & -7 \\ & & 3 & 0 & 0 & 6 \\ \hline & 1 & 0 & 0 & 2 & -1 \end{array}$$

The remainder is -1 , and $P(3) = 81 - 81 + 6 - 7 = -1$. They agree; Volume II proves this always happens (the Remainder Theorem).

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|--|--|
| D7.1 Degree of $4x^5 - x^2 + 1$ | D7.13 $(2y - 7)(2y + 7)$ |
| D7.2 Leading coefficient of $3 - 2x + 7x^3$ | D7.14 $(x + 1)^3$ |
| D7.3 $(2x + 3) + (5x - 1)$ | D7.15 $21 \cdot 19$ |
| D7.4 $(x^2 + 4) - (x^2 - 2x)$ | D7.16 101^2 |
| D7.5 $3x(x - 2)$ | D7.17 $\frac{6x^3 - 4x}{2x}$ |
| D7.6 $(x + 1)(x + 5)$ | D7.18 $(x^2 + 5x + 6) \div (x + 2)$ |
| D7.7 $(x - 3)(x + 4)$ | D7.19 $(x^2 - 9) \div (x - 3)$ |
| D7.8 $(2x + 1)(x - 1)$ | D7.20 $(x^2 + 2x + 5) \div (x + 1)$ |
| D7.9 $(x + 4)^2$ | D7.21 $(x^3 - 27) \div (x - 3)$ |
| D7.10 $(x - 5)^2$ | D7.22 $P(2)$ for $P(x) = x^3 - x - 1$ |
| D7.11 $(3x + 1)^2$ | D7.23 Remainder of $(x^2 + 1) \div (x - 1)$ |
| D7.12 $(x + 6)(x - 6)$ | D7.24 Degree of $(x^2 + 1)(x^3 - x)$ |

ANSWERS TO SKILL DRILLS

D7.1 5. D7.2 7. D7.3 $7x + 2$. D7.4 $2x + 4$. D7.5 $3x^2 - 6x$. D7.6 $x^2 + 6x + 5$.
D7.7 $x^2 + x - 12$. D7.8 $2x^2 - x - 1$. D7.9 $x^2 + 8x + 16$. D7.10 $x^2 - 10x + 25$.
D7.11 $9x^2 + 6x + 1$. D7.12 $x^2 - 36$. D7.13 $4y^2 - 49$. D7.14 $x^3 + 3x^2 + 3x + 1$.
D7.15 399. D7.16 10201. D7.17 $3x^2 - 2$. D7.18 $x + 3$. D7.19 $x + 3$. D7.20
 $x + 1$, remainder 4. D7.21 $x^2 + 3x + 9$. D7.22 5. D7.23 2. D7.24 5.

EXERCISES**COMPUTATION**

- 7.1** Write $7 - 3x^2 + x^4 - x$ in standard form, and give its degree and leading coefficient.
- 7.2** Classify by number of terms and degree: $5x^3$; $2x - 9$; $x^2 + x + 1$.
- 7.3** For $P(x) = 2x^3 - x^2 + 3$, find $P(-2)$ and $P(\frac{1}{2})$.
- 7.4** Add: $(5x^2 - 3x + 8) + (-2x^2 + 7x - 10)$.
- 7.5** Subtract: $(6a^3 - a + 4) - (2a^3 + 3a^2 - a - 1)$.
- 7.6** Simplify $(x^2 - 4xy + y^2) - (3x^2 + xy - 2y^2) + (x^2 - y^2)$.
- 7.7** Multiply: $-3x^2(4x^3 - 2x + 5)$.
- 7.8** Multiply: $(x + 6)(x - 2)$.
- 7.9** Multiply: $(3y - 4)(2y + 5)$.
- 7.10** Multiply: $(5 - 2t)(3 - t)$.
- 7.11** Multiply: $(2x - 1)(x^2 - 3x + 4)$.
- 7.12** Multiply: $(a + b)(a^2 - ab + b^2)$.
- 7.13** Multiply: $(x^2 + x + 1)(x^2 - x + 1)$.
- 7.14** Expand $(x + 7)^2$.
- 7.15** Expand $(4y - 3)^2$.
- 7.16** Expand $(2a + 5b)^2$.
- 7.17** Multiply: $(x - 9)(x + 9)$.
- 7.18** Multiply: $(3m^2 - 2n)(3m^2 + 2n)$.

7.19 Expand $(x + 2)^3$.

7.20 Expand $(2x - 1)^3$.

7.21 Multiply: $[(x + y) + 3][(x + y) - 3]$.

7.22 Compute mentally with special products: 102^2 , 98^2 , and $61 \cdot 59$.

7.23 Divide: $\frac{15x^4 - 10x^3 + 5x^2}{5x^2}$.

7.24 Divide: $\frac{8a^3b^2 - 12a^2b^3 + 4ab}{4ab}$.

7.25 Divide: $(x^2 + 7x + 12) \div (x + 3)$.

7.26 Divide: $(2x^2 - x - 10) \div (x + 2)$.

7.27 Divide: $(3x^3 - 5x^2 + 2x - 8) \div (x - 2)$.

7.28 Divide, giving the quotient and remainder: $(x^3 + 2x - 5) \div (x - 1)$.

7.29 Divide: $(x^4 - 16) \div (x - 2)$.

7.30 Divide, giving the quotient and remainder: $(6x^3 + x^2 - 4x + 5) \div (2x + 3)$.

7.31 Divide: $(x^3 - 1) \div (x - 1)$.

7.32 A rectangle has sides $2x + 3$ and $x - 1$. Write its area and perimeter as polynomials.

7.33 A square of side x is surrounded by a border 2 units wide. Write the area of the border as a polynomial.

7.34 The price at which q items can be sold is $p = 50 - 2q$. Write the revenue $R = pq$ as a polynomial, and find R when $q = 10$.

CONCEPTS AND PROOF

7.35 Prove that $(a + b)^2 = a^2 + 2ab + b^2$ using the distributive property, and explain Fig. 7.1.

7.36 Explain why the product of a polynomial of degree m and one of degree n has degree $m + n$.

7.37 Is the sum of two polynomials of degree 3 always of degree 3? Explain.

7.38 Explain how to check the result of a polynomial division. Use your method to verify $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$.

- 7.39** Show that $(n + 1)^2 - n^2 = 2n + 1$, and conclude that the differences between consecutive perfect squares are the odd numbers.
- 7.40** Use the difference of squares to show that if two numbers have sum 100, their product is at most 2500.

CHALLENGE

- 7.41** Expand $(x + y + z)^2$.
- 7.42** Find the remainder when $x^{100} - 1$ is divided by $x + 1$, without dividing.
- 7.43** Find k so that dividing $x^3 - kx^2 + 2x - 3$ by $x - 3$ leaves remainder 0.
- 7.44** Show that $(x^2 + y^2)(a^2 + b^2) = (ax - by)^2 + (ay + bx)^2$, and use it to write $13 \cdot 17 = 221$ as a sum of two squares.

CHAPTER 8

FACTORING

LEARNING OBJECTIVES

- Factor out the greatest common factor, and factor by grouping.
- Factor trinomials $x^2 + bx + c$ and $ax^2 + bx + c$.
- Recognize and factor differences of squares, perfect square trinomials, and sums and differences of cubes.
- Factor polynomials completely using a systematic strategy.
- Solve polynomial equations with the zero product property, and apply them.

Multiplying polynomials turns a product into a sum; *factoring* turns a sum back into a product. It is the most used skill in algebra, because products have a property that sums do not: a product is zero only when one of its factors is zero. That fact turns a hard equation into several easy ones.

8.1 COMMON FACTORS AND GROUPING

A polynomial is *factored* when it is written as a product. The first step in every factoring problem is to look for the *greatest common factor* (GCF) of the terms: the largest monomial that divides every term. Take the GCF of the coefficients and, for each variable, its lowest power that appears in every term.

Example 8.1 (The greatest common factor). Factor $12x^3y^2 - 18x^2y^4 + 6x^2y$.

Solution. The GCF of 12, 18, 6 is 6; the lowest powers are x^2 and y . So the GCF is $6x^2y$, and dividing each term by it: $12x^3y^2 - 18x^2y^4 + 6x^2y = 6x^2y(2xy - 3y^3 + 1)$. The +1 is essential: $\frac{6x^2y}{6x^2y} = 1$, not 0.

A common factor can itself be a binomial. Four-term polynomials often factor by

grouping: factor pairs of terms, then factor out the common binomial.

Example 8.2 (Grouping). Factor $x^3 - 4x^2 + 3x - 12$.

Solution. $x^3 - 4x^2 + 3x - 12 = x^2(x - 4) + 3(x - 4) = (x^2 + 3)(x - 4)$.

Check by multiplying: $x^3 - 4x^2 + 3x - 12$.

8.2 FACTORING TRINOMIALS

Since $(x + p)(x + q) = x^2 + (p + q)x + pq$, factoring $x^2 + bx + c$ means finding two integers with product c and sum b .

Example 8.3 (Trinomials with leading coefficient 1). (a) $x^2 + 7x + 12$: the numbers with product 12 and sum 7 are 3 and 4, so $x^2 + 7x + 12 = (x + 3)(x + 4)$. (b) $x^2 - 2x - 15$: product -15 , sum -2 : the numbers are -5 and 3, so $x^2 - 2x - 15 = (x - 5)(x + 3)$.

The signs tell you where to look. If $c > 0$, the two numbers have the same sign as b . If $c < 0$, they have opposite signs, and the one with larger absolute value has the sign of b .

For $ax^2 + bx + c$ with $a \neq 1$, the *ac method* reduces the problem to grouping: find two numbers with product ac and sum b , use them to split the middle term, and factor by grouping.

Example 8.4 (The *ac method*). Factor $6x^2 + 11x - 10$.

Solution. $ac = -60$. Two numbers with product -60 and sum 11 are 15 and -4 . Split the middle term and group:

$$6x^2 + 15x - 4x - 10 = 3x(2x + 5) - 2(2x + 5) = (3x - 2)(2x + 5).$$

COMMON ERROR (Check by multiplying). A proposed factorization can always be checked by multiplying it out, and this takes seconds. Sign errors in the middle term are the most common mistake: $(3x + 2)(2x - 5) = 6x^2 - 11x - 10$, which differs from the target only in the sign of $11x$.

8.3 SPECIAL PATTERNS

Reading the special products of Theorem 7.5 backward, and adding the cube formulas, gives the patterns below.

Theorem 8.5 (Factoring patterns).

$$\begin{array}{ll} a^2 - b^2 = (a - b)(a + b) & \text{difference of squares} \\ a^2 \pm 2ab + b^2 = (a \pm b)^2 & \text{perfect square trinomial} \\ a^3 - b^3 = (a - b)(a^2 + ab + b^2) & \text{difference of cubes} \\ a^3 + b^3 = (a + b)(a^2 - ab + b^2) & \text{sum of cubes} \end{array}$$

Each pattern is verified by multiplying out the right side. For instance, $(a - b)(a^2 + ab + b^2) = a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3$.

Example 8.6 (Using the patterns). (a) $49x^2 - 25y^2 = (7x)^2 - (5y)^2 = (7x - 5y)(7x + 5y)$. (b) $4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + 3^2 = (2x - 3)^2$. (c) $x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$. (d) $27a^3 + 1 = (3a)^3 + 1^3 = (3a + 1)(9a^2 - 3a + 1)$.

COMMON ERROR (The sum of two squares). $a^2 + b^2$ does not factor using real numbers: $(a + b)^2 = a^2 + 2ab + b^2$, not $a^2 + b^2$. So $x^2 + 4$ is *prime*, meaning it cannot be factored with integer (or even real) coefficients. Only the *difference* of squares factors.

8.4 FACTORING COMPLETELY

A polynomial is *factored completely* when no factor can be factored further. The following order of steps works for every problem in this book.

Definition 8.7 (A factoring strategy). (1) Factor out the GCF (including -1 if the leading coefficient is negative).

- (2) Count the terms. Two terms: look for a difference of squares or a sum or difference of cubes. Three terms: look for a perfect square, then try trinomial methods. Four terms: try grouping.
- (3) Check whether each factor can be factored again.
- (4) Check the result by multiplying.

Example 8.8 (Factoring completely). (a) $2x^4 - 32 = 2(x^4 - 16) = 2(x^2 - 4)(x^2 + 4) = 2(x - 2)(x + 2)(x^2 + 4)$. The difference of squares $x^2 - 4$ factors again; $x^2 + 4$ does not.

(b) $3x^3 - 12x^2 - 15x = 3x(x^2 - 4x - 5) = 3x(x - 5)(x + 1)$.

(c) $x^4 - 5x^2 + 4$: treat x^2 as the variable, $(x^2 - 1)(x^2 - 4)$, then factor both: $(x - 1)(x + 1)(x - 2)(x + 2)$.

8.5 SOLVING EQUATIONS BY FACTORING

Theorem 8.9 (Zero Product Property). If $ab = 0$, then $a = 0$ or $b = 0$.

Proof. Suppose $ab = 0$ and $a \neq 0$. Multiply both sides by $\frac{1}{a}$: $b = \frac{1}{a} \cdot 0 = 0$. So if a is not zero, b must be. ■

To solve a polynomial equation by factoring: move every term to one side so the other side is 0, factor, and set each factor equal to 0.

Example 8.10 (Solving by factoring). Solve (a) $x^2 - 5x - 14 = 0$, (b) $2x^2 = 7x + 4$, (c) $x^3 = 9x$.

Solution. (a) $(x - 7)(x + 2) = 0$, so $x = 7$ or $x = -2$. (b) $2x^2 - 7x - 4 = 0$, so $(2x + 1)(x - 4) = 0$ and $x = -\frac{1}{2}$ or $x = 4$. (c) $x^3 - 9x = 0$, so $x(x - 3)(x + 3) = 0$ and $x = 0, 3$ or -3 .

COMMON ERROR (Zero on one side, and don't divide by the variable). The property needs a product equal to zero. From $x(x - 3) = 10$ nothing follows about the factors; first rewrite it as $x^2 - 3x - 10 = 0$. And in (c) above, dividing both sides of $x^3 = 9x$ by x would lose the solution $x = 0$.

The solutions of $P(x) = 0$ are the x -intercepts of the graph of $y = P(x)$, so factoring also locates where a graph crosses the x -axis (Fig. 8.1).

Example 8.11 (An area problem). A rectangle's length is 3 m more than its width, and its area is 40 m^2 . Find its dimensions.

Solution. $w(w + 3) = 40$, so $w^2 + 3w - 40 = 0$ and $(w + 8)(w - 5) = 0$. A width cannot be negative, so $w = 5$ and the rectangle is 5 m by 8 m. The equation has two solutions, but only one answers the question; always

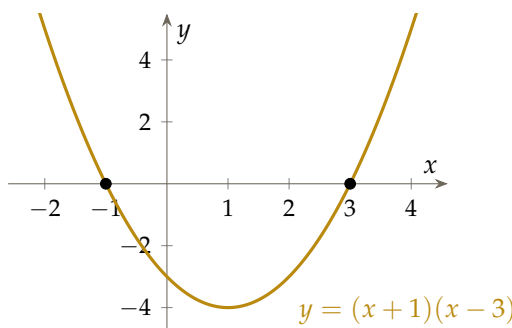


Figure 8.1: $y = x^2 - 2x - 3 = (x + 1)(x - 3)$ crosses the x -axis at the zeros of its factors, $x = -1$ and $x = 3$.

check solutions against the situation.

LOOKING AHEAD. Not every quadratic factors over the integers; Chapter 9 solves them all with the quadratic formula. Factoring is also the key step in simplifying the rational expressions of Chapter 10, and in Volume II the Factor Theorem connects factors and zeros for polynomials of every degree.

8.6 WORKED PROBLEMS

These problems combine the factoring patterns and use factoring to solve equations. Always factor out the GCF first.

PROBLEM 8.1 GCF first

Factor $18x^3 - 33x^2 - 6x$ completely.

Solution. The GCF is $3x$: $3x(6x^2 - 11x - 2)$. For the trinomial, $ac = -12$, and -12 and 1 have product -12 and sum -11 . So $6x^2 - 12x + x - 2 = 6x(x - 2) + 1(x - 2) = (6x + 1)(x - 2)$. Altogether, $18x^3 - 33x^2 - 6x = 3x(6x + 1)(x - 2)$.

PROBLEM 8.2 Three terms that make a square

Factor $x^2 - 10x + 25 - 9y^2$.

Solution. The first three terms form a perfect square: $(x - 5)^2 - 9y^2$. This is a difference of squares with $a = x - 5$ and $b = 3y$: $(x - 5 - 3y)(x - 5 + 3y)$.

Grouping the terms in pairs would not have worked.

PROBLEM 8.3 A difference of sixth powers

Factor $x^6 - 64$ completely.

Solution. Treat it as a difference of squares first: $x^6 - 64 = (x^3 - 8)(x^3 + 8)$. Then use the cube patterns:

$$x^6 - 64 = (x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4).$$

The two quadratic factors do not factor further. (Starting with a difference of cubes, $(x^2)^3 - 4^3$, also works but needs an extra step to finish.)

PROBLEM 8.4 Factoring by substitution

Factor $(x + 1)^2 - 5(x + 1) + 6$.

Solution. Let $u = x + 1$. The expression is $u^2 - 5u + 6 = (u - 2)(u - 3)$. Substituting back: $(x + 1 - 2)(x + 1 - 3) = (x - 1)(x - 2)$. Check at $x = 0$: the original is $1 - 5 + 6 = 2$, and $(-1)(-2) = 2$.

PROBLEM 8.5 A cubic equation

Solve $3x^3 + 2x^2 = 12x + 8$.

Solution. Move everything to one side: $3x^3 + 2x^2 - 12x - 8 = 0$. Group: $x^2(3x + 2) - 4(3x + 2) = (x^2 - 4)(3x + 2) = (x - 2)(x + 2)(3x + 2)$. So $x = 2$, $x = -2$, or $x = -\frac{2}{3}$.

PROBLEM 8.6 Designing a box

An open box is made from a square sheet of metal by cutting 3-inch squares from each corner and folding up the sides. The box must hold 300 in^3 . How large a sheet is needed?

Solution. If the sheet has side s , the base is $(s - 6)$ by $(s - 6)$ and the height is 3, so $3(s - 6)^2 = 300$ and $(s - 6)^2 - 100 = 0$. As a difference of squares, $(s - 6 - 10)(s - 6 + 10) = (s - 16)(s + 4) = 0$. The side must be positive, so $s = 16$ inches. Check: $3 \cdot 10^2 = 300$.

PROBLEM 8.7 An object thrown from a roof

An object thrown upward from a 160-foot roof has height $h = -16t^2 + 48t + 160$ after t seconds. When does it hit the ground, and when is it 192 feet high?

Solution. Ground: $-16(t^2 - 3t - 10) = -16(t - 5)(t + 2) = 0$, so $t = 5$ seconds; the solution $t = -2$ is before the throw. Height 192: $-16t^2 + 48t + 160 = 192$ gives $-16t^2 + 48t - 32 = 0$, that is, $t^2 - 3t + 2 = (t - 1)(t - 2) = 0$. The object is at 192 ft twice: at $t = 1$ s on the way up and at $t = 2$ s on the way down.

SKILL DRILLS

Factor completely, or solve. Answers follow the list.

D8.1 $6x + 9$

D8.2 $x^2 - 7x$

D8.3 $4a^2b - 6ab^2$

D8.4 $x^2 + 5x + 6$

D8.5 $x^2 - 6x + 8$

D8.6 $x^2 + x - 12$

D8.7 $x^2 - 4x - 21$

D8.8 $2x^2 + 5x + 2$

D8.9 $3x^2 - x - 2$

D8.10 $x^2 - 49$

D8.11 $4y^2 - 1$

D8.12 $x^2 - 12x + 36$

D8.13 $9x^2 + 6x + 1$

D8.14 $x^3 - 1$

D8.15 $y^3 + 8$

D8.16 $2x^2 - 18$

D8.17 $x^3 - x$

D8.18 $x^2 + 2x + xy + 2y$

D8.19 Solve $(x - 4)(x + 1) = 0$

D8.20 Solve $x^2 - 9 = 0$

D8.21 Solve $x^2 + 3x = 0$

D8.22 Solve $x^2 - 8x + 15 = 0$

D8.23 Solve $2x^2 - x - 1 = 0$

D8.24 Solve $x^2 = 4x - 4$

ANSWERS TO SKILL DRILLS

D8.1 $3(2x + 3)$. D8.2 $x(x - 7)$. D8.3 $2ab(2a - 3b)$. D8.4 $(x + 2)(x + 3)$. D8.5 $(x - 4)(x - 2)$. D8.6 $(x - 3)(x + 4)$. D8.7 $(x - 7)(x + 3)$. D8.8 $(x + 2)(2x + 1)$. D8.9 $(x - 1)(3x + 2)$. D8.10 $(x - 7)(x + 7)$. D8.11 $(2y - 1)(2y + 1)$. D8.12 $(x - 6)^2$. D8.13 $(3x + 1)^2$. D8.14 $(x - 1)(x^2 + x + 1)$. D8.15

$(y+2)(y^2-2y+4)$. D8.16 $2(x-3)(x+3)$. D8.17 $x(x-1)(x+1)$. D8.18 $(x+2)(x+y)$. D8.19 $x = -1, 4$. D8.20 $x = -3, 3$. D8.21 $x = -3, 0$. D8.22 $x = 3, 5$. D8.23 $x = -\frac{1}{2}, 1$. D8.24 $x = 2$.

EXERCISES

COMPUTATION

- 8.1** Factor $8x^3 - 12x^2$.
- 8.2** Factor $15a^2b - 25ab^2 + 5ab$.
- 8.3** Factor $x(x-4) + 7(x-4)$.
- 8.4** Factor by grouping: $x^3 + 2x^2 + 5x + 10$.
- 8.5** Factor by grouping: $ab - 3a + 2b - 6$.
- 8.6** Factor by grouping: $6x^3 - 9x^2 - 4x + 6$.
- 8.7** Factor $x^2 + 9x + 20$.
- 8.8** Factor $x^2 - 10x + 21$.
- 8.9** Factor $x^2 + 3x - 28$.
- 8.10** Factor $y^2 - y - 42$.
- 8.11** Factor $x^2 - 8xy + 15y^2$.
- 8.12** Factor $2x^2 + 7x + 3$.
- 8.13** Factor $3x^2 - 10x + 8$.
- 8.14** Factor $6x^2 + x - 15$.
- 8.15** Factor $10t^2 - 13t - 3$.
- 8.16** Factor $4x^2 + 4x - 15$.
- 8.17** Factor $x^2 - 64$.
- 8.18** Factor $9a^2 - 49b^2$.
- 8.19** Factor completely: $x^4 - 81$.
- 8.20** Factor $x^2 + 14x + 49$.

- 8.21** Factor $25y^2 - 30y + 9$.
- 8.22** Factor $x^3 + 27$.
- 8.23** Factor $8m^3 - 125$.
- 8.24** Factor completely: $3x^2 - 75$.
- 8.25** Factor completely: $2x^3 + 8x^2 - 42x$.
- 8.26** Factor completely: $x^4 - 5x^2 + 4$.
- 8.27** Factor completely: $5x^4y - 5y$.
- 8.28** Factor $(x + 1)^2 - 16$.
- 8.29** Factor $x^2 + 6x + 9 - y^2$.
- 8.30** Can $x^2 + 4$ be factored over the integers? Can $x^2 - 4$?
- 8.31** Solve $x^2 - 5x - 14 = 0$.
- 8.32** Solve $x^2 = 6x$.
- 8.33** Solve $2x^2 + 5x = 3$.
- 8.34** Solve $4x^2 - 9 = 0$.
- 8.35** Solve $x(x + 2) = 15$.
- 8.36** Solve $x^3 - 4x^2 - 12x = 0$.
- 8.37** Solve $(x - 2)(x + 3) = 6$.
- 8.38** The product of two consecutive positive integers is 156. Find them.
- 8.39** A rectangle's length is 4 cm more than its width, and its area is 96 cm^2 . Find its dimensions.
- 8.40** A ball's height after t seconds is $h = -16t^2 + 32t + 48$ feet. When does it hit the ground?
- 8.41** The legs of a right triangle differ by 7 cm, and its hypotenuse is 13 cm. Find the legs.

CONCEPTS AND PROOF

- 8.42** Prove the zero product property.

- 8.43** A student solves $x(x - 3) = 10$ by writing $x = 10$ or $x - 3 = 10$. Explain the error, and solve the equation correctly.
- 8.44** Verify the difference of cubes formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ by multiplying.
- 8.45** Explain why $x^2 + 1$ cannot be written as a product of two factors $(x - r)(x - s)$ with real numbers r and s .
- 8.46** Find every integer b for which $x^2 + bx + 12$ can be factored over the integers.
- 8.47** Write a quadratic equation, in the form $x^2 + bx + c = 0$, whose solutions are 3 and -5 .

CHALLENGE

- 8.48** Factor $x^4 + x^2 + 1$. *Hint:* add and subtract x^2 to make a difference of squares.
- 8.49** Show that $n^3 - n$ is divisible by 6 for every integer n . *Hint:* factor it.
- 8.50** Solve $x^4 - 13x^2 + 36 = 0$.

CHAPTER 9

QUADRATIC EQUATIONS AND FUNCTIONS

LEARNING OBJECTIVES

- Solve quadratic equations with the square root property and by completing the square.
- Derive and apply the quadratic formula, and use the discriminant to count solutions.
- Graph quadratic functions using vertex form, and find vertices, axes, and intercepts.
- Solve maximum and minimum problems, and systems of a line and a parabola.
- Solve quadratic inequalities.

A *quadratic equation* is one that can be written as $ax^2 + bx + c = 0$ with $a \neq 0$. Factoring solves some of them, but most quadratics, such as $x^2 - 2x - 1 = 0$, do not factor over the integers. This chapter develops a method that solves every quadratic, and then studies the graphs of quadratic functions, the parabolas, which describe everything from thrown balls to maximum profits.

9.1 SQUARE ROOTS AND COMPLETING THE SQUARE

Every positive number d has two square roots, $\sqrt{d}$ and $-\sqrt{d}$, where $\sqrt{d}$ denotes the positive one. For example, the square roots of 49 are 7 and -7 . Square roots of products can be split: $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for $a, b \geq 0$, so $\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$. Chapter 11 treats radicals in full; this much is enough for now.

Theorem 9.1 (Square Root Property). If $X^2 = d$ with $d \geq 0$, then $X = \sqrt{d}$ or $X = -\sqrt{d}$, written $X = \pm\sqrt{d}$. If $d < 0$, the equation has no real solution.

Example 9.2 (The square root property). (a) $2x^2 - 50 = 0$ gives $x^2 = 25$, so $x = \pm 5$. (b) $(x - 3)^2 = 7$ gives $x - 3 = \pm\sqrt{7}$, so $x = 3 \pm \sqrt{7}$.

COMMON ERROR (Both roots). From $x^2 = 25$, the solutions are 5 and -5 . Writing only $x = \sqrt{25} = 5$ loses half the answer; $\sqrt{25}$ is just the positive one.

Any quadratic can be rewritten so the square root property applies. The idea is to make the left side a perfect square. Since $(x + k)^2 = x^2 + 2kx + k^2$, the expression $x^2 + bx$ becomes a perfect square when $\left(\frac{b}{2}\right)^2$ is added:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

This is called *completing the square*.

Example 9.3 (Completing the square). Solve (a) $x^2 + 6x - 7 = 0$ and (b) $2x^2 - 8x + 3 = 0$.

Solution. (a) $x^2 + 6x = 7$. Half of 6 is 3, and $3^2 = 9$; add it to both sides: $x^2 + 6x + 9 = 16$, so $(x + 3)^2 = 16$, $x + 3 = \pm 4$, and $x = 1$ or $x = -7$.

(b) Divide by 2 first so the leading coefficient is 1: $x^2 - 4x = -\frac{3}{2}$. Add $(-2)^2 = 4$: $(x - 2)^2 = \frac{5}{2}$. So $x = 2 \pm \sqrt{\frac{5}{2}} = 2 \pm \frac{\sqrt{10}}{2}$.

9.2 THE QUADRATIC FORMULA

Completing the square on the general equation gives a formula for every quadratic at once.

Theorem 9.4 (Quadratic Formula). The solutions of $ax^2 + bx + c = 0$, where $a \neq 0$ and $b^2 - 4ac \geq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof. Divide by a and move the constant: $x^2 + \frac{b}{a}x = -\frac{c}{a}$. Add $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ to both sides:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

If $b^2 - 4ac \geq 0$, the square root property gives $x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ (since $\sqrt{4a^2} = 2|a|$, and the $\pm$ absorbs the sign of a). Subtracting $\frac{b}{2a}$ gives the formula. ■

Example 9.5 (Using the formula). Solve $3x^2 - 5x - 1 = 0$.

Solution. $a = 3$, $b = -5$, $c = -1$. Then $b^2 - 4ac = 25 + 12 = 37$, so $x = \frac{5 \pm \sqrt{37}}{6}$, about 1.85 and -0.18 .

COMMON ERROR (Signs in the formula). With $b = -5$, the formula begins $-b = +5$, and $b^2 = (-5)^2 = +25$. Write out a , b , c with their signs before substituting. Also, the fraction bar extends under the whole numerator: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, not $-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

The number $b^2 - 4ac$ under the radical is the *discriminant*. It tells how many real solutions there are without solving.

Theorem 9.6 (The discriminant). For $ax^2 + bx + c = 0$ with discriminant $D = b^2 - 4ac$: if $D > 0$ there are two real solutions; if $D = 0$ there is exactly one, $x = -\frac{b}{2a}$; if $D < 0$ there are no real solutions.

Example 9.7 (No real solutions). $x^2 - 4x + 13 = 0$ has $D = 16 - 52 = -36 < 0$, so it has no real solutions. (In Chapter 11 we meet the complex numbers, in which it does have solutions, $2 \pm 3i$.)

Which method should be used? If the equation factors easily, factor. If it has the form $X^2 = d$, use the square root property. Otherwise use the quadratic formula, which always works.

9.3 GRAPHS OF QUADRATIC FUNCTIONS

The graph of a *quadratic function* $f(x) = ax^2 + bx + c$ ($a \neq 0$) is a *parabola*. Completing the square puts it in a form that shows the graph directly.

Theorem 9.8 (Vertex form). Every quadratic function can be written as $f(x) = a(x - h)^2 + k$. Its graph is the parabola $y = ax^2$ shifted so that its vertex is at (h, k) . It opens upward if $a > 0$ and downward if $a < 0$, and it is symmetric about its axis, the vertical line $x = h$. In terms of the coefficients, $h = -\frac{b}{2a}$ and $k = f(h)$.

When $a > 0$, the term $a(x - h)^2$ is never negative and is zero only at $x = h$, so k is the minimum value of f . When $a < 0$, k is the maximum value.

Example 9.9 (Graphing a parabola). Graph $f(x) = 2x^2 - 8x + 5$.

Solution. Complete the square on the x -terms: $f(x) = 2(x^2 - 4x) + 5 = 2(x^2 - 4x + 4) - 8 + 5 = 2(x - 2)^2 - 3$. (Adding 4 inside the parentheses adds $2 \cdot 4 = 8$, so 8 is subtracted outside.) The vertex is $(2, -3)$, the axis is $x = 2$, and the parabola opens upward, so the minimum value is -3 . The y -intercept is $f(0) = 5$, and by symmetry $(4, 5)$ is also on the graph. The x -intercepts solve $2(x - 2)^2 = 3$: $x = 2 \pm \sqrt{\frac{3}{2}} \approx 0.78$ and 3.22 . See Fig. 9.1.

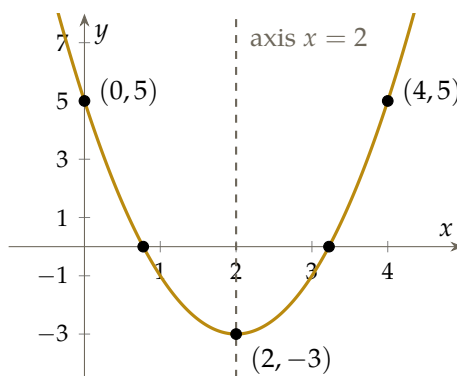


Figure 9.1: $f(x) = 2x^2 - 8x + 5 = 2(x - 2)^2 - 3$: vertex $(2, -3)$, axis $x = 2$, and the symmetric pair of points $(0, 5)$ and $(4, 5)$.

9.4 MAXIMUM AND MINIMUM PROBLEMS

Because the vertex gives the largest or smallest value of a quadratic function, any quantity modeled by a quadratic can be optimized by finding its vertex.

Example 9.10 (Maximum height). A ball's height after t seconds is $h(t) = -16t^2 + 64t + 5$ feet. When does it reach its highest point, and how high is it?

Solution. The vertex is at $t = -\frac{64}{2(-16)} = 2$ seconds, and $h(2) = -64 + 128 + 5 = 69$ feet.

Example 9.11 (Fencing against a wall). A rectangular garden against a wall is fenced on three sides with 100 m of fence. What dimensions give the largest area?

Solution. With width x (the sides perpendicular to the wall), the area is $A(x) = x(100 - 2x) = -2x^2 + 100x$. Its vertex is at $x = -\frac{100}{2(-2)} = 25$, so the best garden is 25 m by 50 m, with area 1250 m^2 .

Example 9.12 (A line and a parabola). Find where the line $y = x + 4$ meets the parabola $y = x^2 - 2x$.

Solution. Setting the expressions for y equal: $x^2 - 2x = x + 4$, so $x^2 - 3x - 4 = 0$ and $(x - 4)(x + 1) = 0$. The points are $(4, 8)$ and $(-1, 3)$. In general, a line meets a parabola in two points, one point, or none, according to the discriminant of the resulting quadratic.

9.5 QUADRATIC INEQUALITIES

To solve an inequality such as $x^2 - x - 6 < 0$, find where the quadratic is zero and then determine its sign between those points. A polynomial can change sign only at a zero, so its sign is constant on each interval between consecutive zeros, and one test value per interval decides it.

Example 9.13 (A sign chart). Solve $x^2 - x - 6 < 0$.

Solution. $x^2 - x - 6 = (x - 3)(x + 2)$ is zero at -2 and 3 . Test one value in each interval:

Interval	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
Test value	-3	0	4
Sign of $(x - 3)(x + 2)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$

The product is negative on $(-2, 3)$, so that is the solution. On the graph,

this is where the parabola lies below the x -axis.

COMMON ERROR (Don't divide by the variable). To solve $x^2 \geq 4x$, do not divide by x ; its sign is unknown. Instead write $x^2 - 4x \geq 0$, that is, $x(x - 4) \geq 0$, which holds for $x \leq 0$ or $x \geq 4$.

LOOKING AHEAD. In Volume II the quadratic formula is extended to negative discriminants using complex numbers, and the vertex form becomes one case of the general theory of transformations of graphs. In Volume III, the vertex of a parabola is found a second way, as the point where the slope of the tangent line is zero; this is the start of optimization with derivatives.

9.6 WORKED PROBLEMS

These problems use the quadratic formula, vertex form, and the discriminant. Before solving, decide which method is shortest.

PROBLEM 9.1 Clearing fractions first

Solve $\frac{x^2}{2} - \frac{x}{3} = 1$.

Solution. Multiply by 6: $3x^2 - 2x - 6 = 0$. The quadratic formula gives $x = \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{2 \pm \sqrt{76}}{6}$. Since $\sqrt{76} = 2\sqrt{19}$, this simplifies to $x = \frac{1 \pm \sqrt{19}}{3}$, about 1.79 and -1.12 .

PROBLEM 9.2 A downward parabola

Find the vertex and x -intercepts of $f(x) = -x^2 - 4x + 1$.

Solution. Factor -1 from the x -terms: $f(x) = -(x^2 + 4x) + 1 = -(x^2 + 4x + 4) + 4 + 1 = -(x + 2)^2 + 5$. Adding 4 inside the parentheses subtracted 4, so 4 is added back outside. The vertex is $(-2, 5)$, a maximum. The intercepts solve $(x + 2)^2 = 5$: $x = -2 \pm \sqrt{5}$, about 0.24 and -4.24 .

PROBLEM 9.3 A parabola through three points

Find the quadratic function whose graph passes through $(0, 3)$, $(1, 2)$ and $(2, 5)$.

Solution. Write $f(x) = ax^2 + bx + c$. The point $(0, 3)$ gives $c = 3$. Then $(1, 2)$ gives $a + b + 3 = 2$, and $(2, 5)$ gives $4a + 2b + 3 = 5$. So $a + b = -1$ and $2a + b = 1$; subtracting, $a = 2$ and $b = -3$. The function is $f(x) = 2x^2 - 3x + 3$.

PROBLEM 9.4 Pricing for maximum revenue

At a price of p dollars, a store sells $q = 1200 - 30p$ units per week. What price maximizes revenue?

Solution. Revenue is price times quantity: $R = p(1200 - 30p) = -30p^2 + 1200p$. The vertex is at $p = -\frac{1200}{2(-30)} = 20$. At \$20 the store sells 600 units, for revenue \$12,000. A higher price loses more in sales than it gains per unit.

PROBLEM 9.5 A projectile in meters

A ball is thrown upward from a height of 1.5 m at 20 m/s. Its height is $h = -4.9t^2 + 20t + 1.5$ meters after t seconds. Find its maximum height and when it lands.

Solution. The vertex is at $t = \frac{20}{9.8} \approx 2.04$ s, where $h \approx -4.9(2.041)^2 + 20(2.041) + 1.5 \approx 21.91$ m. It lands when $h = 0$:

$$t = \frac{-20 \pm \sqrt{400 + 4(4.9)(1.5)}}{2(-4.9)} = \frac{20 \mp \sqrt{429.4}}{9.8}.$$

The positive solution is $t \approx \frac{20 + 20.722}{9.8} \approx 4.16$ s; the negative one is before the throw.

PROBLEM 9.6 Tangent lines from the discriminant

For which k does the line $y = kx$ touch the parabola $y = x^2 + 3$ at exactly one point?

Solution. Setting $x^2 + 3 = kx$ gives $x^2 - kx + 3 = 0$. There is exactly one solution when the discriminant is zero: $k^2 - 12 = 0$, so $k = \pm\sqrt{12} = \pm 2\sqrt{3}$. These are the two lines through the origin tangent to the parabola.

PROBLEM 9.7 When is the business profitable?

A company's weekly profit from producing x hundred units is $P = -x^2 + 40x - 300$ hundred dollars. For which x is the profit positive, and what is the largest profit?

Solution. $P > 0$ means $x^2 - 40x + 300 < 0$, that is, $(x - 10)(x - 30) < 0$. The product is negative between the zeros, so $10 < x < 30$. The vertex is at $x = 20$, where $P = -400 + 800 - 300 = 100$. The largest profit is \$10,000, at 2000 units.

PROBLEM 9.8 An equation quadratic in form

Solve $(x^2 - 3)^2 - 4(x^2 - 3) - 12 = 0$.

Solution. Let $u = x^2 - 3$. Then $u^2 - 4u - 12 = (u - 6)(u + 2) = 0$, so $u = 6$ or $u = -2$. If $x^2 - 3 = 6$, then $x = \pm 3$; if $x^2 - 3 = -2$, then $x = \pm 1$. The four solutions are ± 1 and ± 3 .

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D9.1 Solve $x^2 = 81$

D9.2 Solve $x^2 - 5 = 0$

D9.3 Solve $(x - 1)^2 = 9$

D9.4 Number that completes the square for $x^2 + 8x$

D9.5 Number that completes the square for $x^2 - 3x$

D9.6 Discriminant of $x^2 + 3x + 1$

D9.7 Discriminant of $2x^2 - 4x + 2$

D9.8 Number of real solutions of $x^2 + 2x + 5 = 0$

D9.9 Solve $x^2 + 2x - 3 = 0$

D9.10 Solve $x^2 - 4x + 1 = 0$

D9.11 Solve $x^2 + x - 1 = 0$

D9.12 Solve $2x^2 - 3x - 2 = 0$

D9.13 Solve $x^2 + 4 = 0$

D9.14 Simplify $\sqrt{50}$

D9.15 Simplify $\sqrt{72}$

D9.16 Vertex of $y = (x - 3)^2 + 2$

D9.17 Vertex of $y = -2(x + 1)^2 - 4$

D9.18 x -coordinate of the vertex of $y = x^2 - 6x + 1$

D9.19 Vertex of $y = x^2 + 4x$

D9.20 Does $y = -3x^2 + x$ open up or down?

D9.21 Minimum value of $y = x^2 - 2x + 5$

D9.22 Maximum value of
 $y = -x^2 + 6x$

D9.23 Solve $x^2 - 4 < 0$

D9.24 Solve $x^2 - 9 \geq 0$

ANSWERS TO SKILL DRILLS

D9.1 $x = \pm 9$. D9.2 $x = \pm\sqrt{5}$. D9.3 $x = -2, 4$. D9.4 16. D9.5 $\frac{9}{4}$. D9.6 5.
D9.7 0. D9.8 None. D9.9 $x = -3, 1$. D9.10 $x = 2 \pm \sqrt{3}$. D9.11 $x = \frac{-1 \pm \sqrt{5}}{2}$.
D9.12 $x = -\frac{1}{2}, 2$. D9.13 No real solution. D9.14 $5\sqrt{2}$. D9.15 $6\sqrt{2}$. D9.16
(3, 2). D9.17 $(-1, -4)$. D9.18 3. D9.19 $(-2, -4)$. D9.20 Down. D9.21 4.
D9.22 9. D9.23 $-2 < x < 2$. D9.24 $x \leq -3$ or $x \geq 3$.

EXERCISES

COMPUTATION

9.1 Solve $x^2 = 49$.

9.2 Solve $3x^2 - 48 = 0$.

9.3 Solve $(x + 2)^2 = 25$.

9.4 Solve $(2x - 1)^2 = 8$.

9.5 What number completes the square for $x^2 + 10x$? Write the result as a square.

9.6 What number completes the square for $x^2 - 7x$? Write the result as a square.

9.7 Solve by completing the square: $x^2 - 6x + 4 = 0$.

9.8 Solve by completing the square: $x^2 + 4x - 21 = 0$.

9.9 Solve by completing the square: $2x^2 + 12x - 5 = 0$.

9.10 Solve with the quadratic formula: $x^2 + 3x - 10 = 0$.

9.11 Solve: $2x^2 - 7x + 2 = 0$.

9.12 Solve: $3x^2 + 2x - 2 = 0$.

9.13 Solve: $x^2 - 2x - 1 = 0$.

9.14 Solve: $4x^2 + 4x + 1 = 0$.

9.15 Solve: $x^2 + x + 5 = 0$.

- 9.16** Solve: $0.5x^2 - 1.2x - 0.8 = 0$, and give decimal approximations.
- 9.17** Use the discriminant to find the number of real solutions: (a) $x^2 - 6x + 9 = 0$; (b) $2x^2 + x + 3 = 0$; (c) $x^2 + 5x - 2 = 0$.
- 9.18** Find k so that $x^2 + kx + 16 = 0$ has exactly one solution.
- 9.19** Solve $x(x - 4) = 3$.
- 9.20** Solve $(x + 1)(x - 3) = x$.
- 9.21** Write $f(x) = x^2 - 6x + 5$ in vertex form, and give the vertex.
- 9.22** Write $f(x) = -2x^2 + 8x - 3$ in vertex form, and give its maximum value.
- 9.23** Write $f(x) = 3x^2 + 6x + 1$ in vertex form, and give the vertex.
- 9.24** Graph $f(x) = x^2 - 4x - 5$, labeling the vertex and intercepts.
- 9.25** Graph $f(x) = -x^2 + 2x + 3$, labeling the vertex and intercepts.
- 9.26** Find the range of $f(x) = 2(x + 1)^2 - 7$.
- 9.27** Find the quadratic function with vertex $(2, -1)$ whose graph passes through $(0, 3)$.
- 9.28** Find the maximum value of $f(x) = -x^2 + 10x - 16$.
- 9.29** A ball's height is $h = -16t^2 + 48t + 4$ feet after t seconds. Find its maximum height and when it occurs.
- 9.30** Two numbers have sum 30. What is the largest possible product?
- 9.31** Find the largest area of a rectangle with perimeter 60 m.
- 9.32** A theater sells 300 tickets at \$20 each. For each \$1 increase in price, 10 fewer tickets are sold. What price maximizes revenue, and what is the maximum revenue?
- 9.33** Solve the system $y = x^2, y = 2x + 3$.
- 9.34** Solve the system $y = x^2 - 4, y = -x^2 + 4$.
- 9.35** Find all points where the line $y = 2x - 5$ meets the parabola $y = x^2 - 4x + 4$. What does the answer say about the line?
- 9.36** Solve $x^2 - 5x + 4 < 0$.
- 9.37** Solve $x^2 + 2x - 15 \geq 0$.

9.38 Solve $2x^2 < 3x + 2$.

9.39 Solve $x^2 + 1 > 0$.

9.40 For the ball in Exercise 29, during what time interval is it more than 36 feet high?

9.41 The legs of a right triangle differ by 1 cm and its hypotenuse is 10 cm. Find the legs to two decimal places.

CONCEPTS AND PROOF

9.42 Derive the quadratic formula by completing the square on $ax^2 + bx + c = 0$.

9.43 Show that the solutions r_1, r_2 of $ax^2 + bx + c = 0$ satisfy $r_1 + r_2 = -\frac{b}{a}$ and $r_1 r_2 = \frac{c}{a}$.

9.44 Explain, using vertex form, why the parabola $y = ax^2 + bx + c$ opens upward when $a > 0$.

9.45 Complete the square on $ax^2 + bx + c$ to show that the vertex is at $x = -\frac{b}{2a}$.

9.46 Explain the connection between the discriminant of $ax^2 + bx + c$ and the number of x -intercepts of its graph.

9.47 Explain why $(x - 3)^2 = -4$ has no real solution.

CHALLENGE

9.48 Solve $x - 5\sqrt{x} + 6 = 0$. *Hint:* let $u = \sqrt{x}$.

9.49 Find the positive number x with $x^2 = x + 1$ (the *golden ratio*).

9.50 Find the point on the line $y = 2x + 1$ closest to the origin. *Hint:* minimize the square of the distance.



PART IV

RATIONAL, RADICAL, AND EXPONENTIAL FUNCTIONS

Rational expressions and equations, radicals and rational exponents, and a first look at exponential and logarithmic functions.

CHAPTER 10

RATIONAL EXPRESSIONS AND EQUATIONS

LEARNING OBJECTIVES

- Find the domain of a rational expression and simplify it by canceling common factors.
- Multiply, divide, add, and subtract rational expressions.
- Simplify complex fractions.
- Solve rational equations, and identify extraneous solutions.
- Solve work, motion, and variation problems.

A *rational expression* is a quotient of two polynomials, such as $\frac{x+1}{x^2-4}$. Rational expressions are to polynomials what fractions are to integers, and they follow exactly the same rules. The one new feature is that a variable can make a denominator zero, so every rational expression comes with a domain that must be tracked.

10.1 SIMPLIFYING RATIONAL EXPRESSIONS

Definition 10.1 (Rational expression). A *rational expression* is an expression $\frac{P}{Q}$ where P and Q are polynomials and Q is not the zero polynomial. Its domain is all real numbers except the zeros of Q .

The basic rule of fractions, $\frac{ac}{bc} = \frac{a}{b}$ for $c \neq 0$, says that a common *factor* of the numerator and denominator can be canceled. To simplify, factor both completely and cancel common factors.

Example 10.2 (Simplifying). Simplify $\frac{x^2 - 9}{x^2 + x - 6}$.

Solution. $\frac{(x-3)(x+3)}{(x+3)(x-2)} = \frac{x-3}{x-2}$, for $x \neq -3, 2$. The restriction $x \neq -3$ must be kept: the original expression is undefined there even though the simplified one is not.

Example 10.3 (Opposite factors). Simplify $\frac{2-x}{x^2-4}$.

Solution. $2-x = -(x-2)$, so $\frac{-(x-2)}{(x-2)(x+2)} = -\frac{1}{x+2}$. Factors that differ only in sign cancel to -1 .

COMMON ERROR (Cancel factors, never terms). $\frac{x+5}{5} \neq x+1$ and $\frac{x^2+4}{x+2} \neq x+2$. Only a factor of the *entire* numerator and the *entire* denominator may be canceled. Test with a number: at $x = 5$, $\frac{x+5}{5} = 2$ but $x+1 = 6$.

10.2 PRODUCTS AND QUOTIENTS

Multiply rational expressions as fractions, numerators times numerators and denominators times denominators, and divide by multiplying by the reciprocal. Factor first and cancel before multiplying; it is much easier than multiplying out and factoring afterward.

Example 10.4 (Multiplying and dividing). (a) $\frac{x^2-1}{x^2+2x} \cdot \frac{x+2}{x-1} =$
 $\frac{(x-1)(x+1)}{x(x+2)} \cdot \frac{x+2}{x-1} = \frac{x+1}{x}$.
 (b) $\frac{x^2-4}{3x} \div \frac{x+2}{6x^2} = \frac{(x-2)(x+2)}{3x} \cdot \frac{6x^2}{x+2} = 2x(x-2)$.

10.3 SUMS AND DIFFERENCES

Fractions with the same denominator are added by adding numerators. For different denominators, rewrite each over the *least common denominator* (LCD): factor every denominator, and take each factor the greatest number of times it

appears in any one denominator.

Example 10.5 (Adding). Add $\frac{3}{x-2} + \frac{5}{x+1}$.

Solution. The LCD is $(x-2)(x+1)$. Then

$$\frac{3(x+1)}{(x-2)(x+1)} + \frac{5(x-2)}{(x-2)(x+1)} = \frac{3x+3+5x-10}{(x-2)(x+1)} = \frac{8x-7}{(x-2)(x+1)}.$$

Example 10.6 (Subtracting). Simplify $\frac{x}{x^2-9} - \frac{1}{x-3}$.

Solution. $x^2 - 9 = (x-3)(x+3)$, which is the LCD. Then

$$\frac{x - (x+3)}{(x-3)(x+3)} = \frac{-3}{(x-3)(x+3)}.$$

COMMON ERROR (Subtract the whole numerator). When subtracting, put the second numerator in parentheses: $x - (x+3) = -3$, not $x - x + 3 = 3$.

10.4 COMPLEX FRACTIONS

A *complex fraction* has fractions in its numerator or denominator. The quickest way to simplify one is to multiply its numerator and denominator by the LCD of all the small fractions.

Example 10.7 (A complex fraction). Simplify $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x^2}}$.

Solution. Multiply the top and bottom by x^2 : $\frac{x^2 + x}{x^2 - 1} = \frac{x(x+1)}{(x-1)(x+1)} = \frac{x}{x-1}$.

10.5 RATIONAL EQUATIONS

To solve an equation containing rational expressions, multiply both sides by the LCD. This clears the fractions and leaves a polynomial equation. But multiplying by an expression that can be zero can introduce solutions that do not satisfy the original equation, called *extraneous* solutions. Every solution must be checked against the domain.

Example 10.8 (Clearing denominators). Solve $\frac{1}{x} + \frac{1}{x+2} = \frac{5}{12}$.

Solution. Multiply by $12x(x+2)$: $12(x+2) + 12x = 5x(x+2)$, so $5x^2 - 14x - 24 = 0$ and $(5x+6)(x-4) = 0$. Both $x = 4$ and $x = -\frac{6}{5}$ are in the domain ($x \neq 0, -2$), and both check: $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

Example 10.9 (An extraneous solution). Solve $\frac{x}{x-3} = \frac{3}{x-3} + 2$.

Solution. Multiply by $x-3$: $x = 3 + 2(x-3)$, so $x = 2x-3$ and $x = 3$. But $x = 3$ makes the denominators zero, so it is not in the domain. It is extraneous, and the equation has *no solution*.

10.6 APPLICATIONS

10.6.1 WORK PROBLEMS

If a job takes t hours, then $\frac{1}{t}$ of the job is done each hour. When people or machines work together, their rates add.

Example 10.10 (Working together). One pipe fills a tank in 6 hours and another in 3 hours. How long does it take with both pipes?

Solution. Together they fill $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ of the tank per hour, so the tank takes 2 hours. (The answer must be less than the faster pipe's time, which is a quick check.)

10.6.2 MOTION PROBLEMS

Since time is distance divided by rate, problems that compare travel times lead to rational equations.

Example 10.11 (Against the current). A boat that goes 15 mph in still water travels 36 miles downstream and back in 5 hours. Find the speed of the current.

Solution. With current c , the times are $\frac{36}{15+c}$ and $\frac{36}{15-c}$, so $\frac{36}{15+c} + \frac{36}{15-c} = 5$. Multiplying by $(15+c)(15-c)$: $36(15-c) + 36(15+c) = 5(225 - c^2)$, so $1080 = 1125 - 5c^2$, $c^2 = 9$, and $c = 3$ mph (the current cannot be negative).

10.6.3 VARIATION

Two quantities *vary directly* if $y = kx$ for a constant k , and *vary inversely* if $y = \frac{k}{x}$. In direct variation, doubling x doubles y ; in inverse variation, doubling x halves y (Fig. 10.1).

Example 10.12 (Inverse variation). At constant temperature, the pressure P of a gas varies inversely with its volume V . If $P = 30$ when $V = 4$, find P when $V = 6$.

Solution. $P = \frac{k}{V}$ with $30 = \frac{k}{4}$, so $k = 120$ and $P = \frac{120}{6} = 20$.

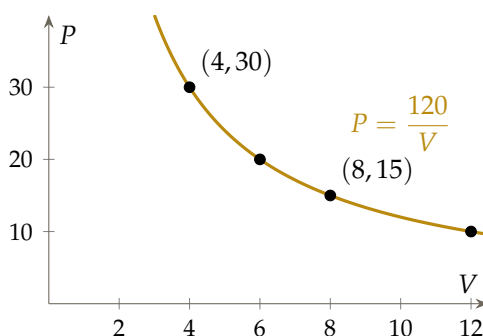


Figure 10.1: Inverse variation: doubling V from 4 to 8 halves P from 30 to 15.

LOOKING AHEAD. In Volume II, rational expressions become *rational functions*, whose graphs have vertical and horizontal asymptotes, and partial fractions reverse the addition of Section 10.3. In Volume III, the simplification $\frac{x^2-9}{x-3} = x+3$ for $x \neq 3$ is exactly how many limits are computed.

10.7 WORKED PROBLEMS

These problems use the arithmetic of rational expressions and rational equations. Factor every denominator first, and check every solution against the domain.

PROBLEM 10.1 A combination that collapses

Simplify $\left(\frac{x}{x-1} - \frac{1}{x+1} \right) \div \frac{x^2+1}{x^2-1}$.

Solution. The difference over the LCD $(x - 1)(x + 1)$ is $\frac{x(x+1) - (x-1)}{(x-1)(x+1)} = \frac{x^2+1}{(x-1)(x+1)}$. Dividing by $\frac{x^2+1}{x^2-1}$ means multiplying by $\frac{(x-1)(x+1)}{x^2+1}$, and everything cancels: the result is 1, for $x \neq \pm 1$.

PROBLEM 10.2 One genuine solution and one extraneous

Solve $1 + \frac{3}{x-2} = \frac{12}{x^2-4}$.

Solution. The LCD is $(x-2)(x+2)$, and the domain excludes ± 2 . Multiplying: $(x^2-4) + 3(x+2) = 12$, so $x^2+3x-10=0$ and $(x+5)(x-2)=0$. The candidate $x=2$ is excluded, so it is extraneous. The only solution is $x=-5$. Check: $1 + \frac{3}{-7} = \frac{4}{7}$ and $\frac{12}{21} = \frac{4}{7}$.

PROBLEM 10.3 Three workers

Working alone, three people can do a job in 4, 6 and 12 hours. How long does it take all three together?

Solution. Their rates add: $\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{1}{2}$ job per hour. Together they take 2 hours.

PROBLEM 10.4 Average speed of a round trip

A driver goes to a city at 60 mph and returns along the same road at 40 mph. What is the average speed for the round trip?

Solution. The answer is not 50 mph. Let the one-way distance be d . The total time is $\frac{d}{60} + \frac{d}{40} = \frac{2d+3d}{120} = \frac{5d}{120}$, so the average speed is total distance over total time: $\frac{2d}{5d/120} = 48$ mph. The driver spends more time at the slower speed, which pulls the average below 50.

PROBLEM 10.5 Combined variation

y varies directly with x and inversely with the square of z . If $y = 12$ when $x = 6$ and $z = 2$, find y when $x = 9$ and $z = 3$.

Solution. $y = \frac{kx}{z^2}$. From the data, $12 = \frac{6k}{4}$, so $k = 8$. Then $y = \frac{8 \cdot 9}{9} = 8$.

PROBLEM 10.6 Solving a formula

Solve $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ for b .

Solution. Isolate the term with b : $\frac{1}{b} = \frac{1}{c} - \frac{1}{a} = \frac{a-c}{ac}$. Taking reciprocals, $b = \frac{ac}{a-c}$, provided $a \neq c$.

PROBLEM 10.7 A lens

A lens with focal length f forms an image at distance d_i of an object at distance d_o , where $\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$. If $f = 10$ cm and $d_o = 30$ cm, find d_i .

Solution. $\frac{1}{d_i} = \frac{1}{10} - \frac{1}{30} = \frac{3-1}{30} = \frac{1}{15}$, so $d_i = 15$ cm.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D10.1 Domain of $\frac{1}{x+4}$

D10.2 Domain of $\frac{x}{x^2-1}$

D10.3 $\frac{6x^2}{9x}$

D10.4 $\frac{x+3}{x^2-9}$

D10.5 $\frac{x^2-4}{x-2}$

D10.6 $\frac{x-1}{1-x}$

D10.7 $\frac{2}{x} \cdot \frac{x^2}{4}$

D10.8 $\frac{3}{x} \div \frac{6}{x^2}$

D10.9 $\frac{1}{x} + \frac{1}{2x}$

D10.10 $\frac{1}{2} + \frac{1}{x}$

D10.11 $\frac{3}{x+1} - \frac{1}{x+1}$

D10.12 $\frac{1}{x-1} + \frac{1}{x+1}$

D10.13 $\frac{x}{x-2} - \frac{2}{x-2}$

D10.14 $\frac{1/2}{1/x}$

D10.15 $\frac{1-\frac{1}{x}}{x-1}$

D10.16 Solve $\frac{6}{x} = 3$

D10.17 Solve $\frac{x}{5} = \frac{4}{10}$

D10.18 Solve $\frac{1}{x} + \frac{1}{3} = \frac{1}{2}$

D10.19 Solve $\frac{3}{x+1} = 1$

D10.20 Solve $\frac{x}{x-1} = \frac{1}{x-1}$

D10.21 Jobs taking 2 h and 3 h, done together

D10.22 $y = kx$ and $y = 10$ when $x = 2$: find k

- D10.23** $y = \frac{k}{x}$ and $y = 4$ when $x = 5$: find y at $x = 10$ **D10.24** Time to travel 150 miles at r mph

ANSWERS TO SKILL DRILLS

D10.1 $x \neq -4$. D10.2 $x \neq \pm 1$. D10.3 $\frac{2x}{3}$. D10.4 $\frac{1}{x-3}$. D10.5 $x + 2$. D10.6 -1 .
D10.7 $\frac{x}{2}$. D10.8 $\frac{x}{2}$. D10.9 $\frac{3}{2x}$. D10.10 $\frac{x+2}{2x}$. D10.11 $\frac{2}{x+1}$. D10.12 $\frac{2x}{x^2-1}$.
D10.13 1. D10.14 $\frac{x}{2}$. D10.15 $\frac{1}{x}$. D10.16 $x = 2$. D10.17 $x = 2$. D10.18 $x = 6$.
D10.19 $x = 2$. D10.20 No solution. D10.21 $\frac{6}{5}$ h. D10.22 5. D10.23 2. D10.24 $\frac{150}{r}$ hours.

EXERCISES

COMPUTATION

- 10.1** Find the domain of $\frac{x+1}{x^2-5x+6}$.
- 10.2** Simplify $\frac{12x^3y}{18xy^4}$.
- 10.3** Simplify $\frac{x^2-16}{x^2+8x+16}$.
- 10.4** Simplify $\frac{3x-6}{x^2-4}$.
- 10.5** Simplify $\frac{x^2-5x+6}{x^2-9}$.
- 10.6** Simplify $\frac{5-x}{x^2-25}$.
- 10.7** Simplify $\frac{x^3-8}{x^2-4}$.
- 10.8** Multiply: $\frac{x^2-x-6}{x^2-4} \cdot \frac{x+2}{x-3}$.
- 10.9** Multiply: $\frac{2x^2+x-1}{x^2-1} \cdot \frac{x+1}{2x-1}$.
- 10.10** Divide: $\frac{x^2-9}{4x} \div \frac{x+3}{2x^2}$.
- 10.11** Divide: $\frac{a^2-ab}{b^2} \div \frac{a^2-b^2}{ab+b^2}$.

10.12 Add: $\frac{2}{x} + \frac{3}{x^2}$.

10.13 Subtract: $\frac{5}{x+2} - \frac{3}{x-1}$.

10.14 Add: $\frac{x}{x-4} + \frac{4}{4-x}$.

10.15 Add: $\frac{3}{x^2-1} + \frac{2}{x+1}$.

10.16 Subtract: $\frac{x}{x^2-x-2} - \frac{2}{x^2-4}$.

10.17 Simplify $1 - \frac{1}{x+1}$.

10.18 Simplify $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x^2}}$.

10.19 Simplify $\frac{\frac{2}{a} - \frac{3}{b}}{\frac{4}{a^2} - \frac{9}{b^2}}$.

10.20 Simplify $\frac{\frac{x}{x+1} - 1}{\frac{x}{x+1} + 1}$.

10.21 Solve $\frac{x}{3} + \frac{x}{4} = 7$.

10.22 Solve $\frac{4}{x} - 1 = \frac{3}{x}$.

10.23 Solve $\frac{2}{x-1} = \frac{3}{x+2}$.

10.24 Solve $\frac{1}{x} + \frac{1}{2x} = \frac{1}{4}$.

10.25 Solve $\frac{x}{x-2} = \frac{2}{x-2} + 3$.

10.26 Solve $\frac{3}{x-1} + 1 = \frac{6}{x^2-1}$.

10.27 Solve $x + \frac{6}{x} = 5$.

10.28 Solve $\frac{1}{x+1} + \frac{1}{x-1} = \frac{4}{3}$.

- 10.29** Solve $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R .
- 10.30** Solve $y = \frac{x+1}{x-2}$ for x .
- 10.31** One pipe fills a pool in 10 hours and another in 15 hours. How long do they take together?
- 10.32** Two painters together paint a room in 4 hours. One alone takes 6 hours. How long does the other take alone?
- 10.33** A tap fills a tub in 3 hours and a drain empties it in 5 hours. With both open, how long does the empty tub take to fill?
- 10.34** A plane with airspeed 180 mph flies 600 miles with the wind in the same time it flies 480 miles against it. Find the wind speed.
- 10.35** A boat with speed 12 mph in still water travels 18 miles upstream and back in 3.2 hours. Find the speed of the current.
- 10.36** The sum of a number and its reciprocal is $\frac{10}{3}$. Find the number.
- 10.37** y varies directly with x , and $y = 15$ when $x = 6$. Find y when $x = 10$.
- 10.38** y varies inversely with x , and $y = 8$ when $x = 3$. Find y when $x = 12$.
- 10.39** The distance an object falls varies directly with the square of the time. It falls 64 ft in 2 s. How far does it fall in 5 s?
- 10.40** Light intensity varies inversely with the square of the distance from the source. It is 90 units at 2 m. Find it at 6 m.

CONCEPTS AND PROOF

- 10.41** Explain why $\frac{x+5}{5} \neq x+1$, using a numerical example and the rule for canceling.
- 10.42** Explain why $\frac{x^2-9}{x-3}$ and $x+3$ are not equal for every x , though they agree wherever both are defined.
- 10.43** Why must the solutions of a rational equation be checked? Give an example of an equation with an extraneous solution.
- 10.44** Prove that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, and explain why the LCD is usually a better choice than bd .

- 10.45** If y varies inversely with x , what happens to y when x is doubled? When x is tripled?

CHALLENGE

10.46 Simplify $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{x}}}$.

10.47 Find A and B with $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$, and use the result to find $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{99 \cdot 100}$.

- 10.48** Two pipes together fill a tank in 6 hours. Alone, one takes 5 hours longer than the other. How long does each take alone?

CHAPTER 11

RADICALS AND RATIONAL EXPONENTS

LEARNING OBJECTIVES

- Evaluate n th roots, and simplify radical expressions.
- Add, subtract, multiply, and divide radical expressions, and rationalize denominators.
- Interpret and compute with rational exponents.
- Solve radical equations and identify extraneous solutions.
- Compute with complex numbers, and solve every quadratic equation.

Squaring a number has an inverse operation, taking a square root, and the same is true for every power. This chapter studies roots, writes them as fractional exponents so that the laws of Chapter 2 apply to them, and then meets a number system larger than the reals, the complex numbers, in which every quadratic equation has solutions.

11.1 ROOTS

Definition 11.1 (n th roots). Let $n \geq 2$ be an integer. A number b is an n th root of a if $b^n = a$. The *principal n th root* $\sqrt[n]{a}$ is defined as follows.

- If n is even and $a \geq 0$, $\sqrt[n]{a}$ is the nonnegative n th root. If n is even and $a < 0$, $\sqrt[n]{a}$ is not a real number.
- If n is odd, $\sqrt[n]{a}$ is the unique real n th root, which has the same sign as a .

The number a is the *radicand* and n the *index*; $\sqrt{a}$ means $\sqrt[2]{a}$.

So $\sqrt{16} = 4$ (not -4), $\sqrt[3]{-27} = -3$, $\sqrt[4]{81} = 3$, and $\sqrt[4]{-16}$ is not real, since no real

number raised to the fourth power is negative.

Theorem 11.2 (Roots of powers). For every real x : $\sqrt{x^2} = |x|$ and $\sqrt[3]{x^3} = x$. More generally, $\sqrt[n]{x^n} = |x|$ if n is even and x if n is odd.

The absolute value is needed because $\sqrt{}$ is never negative: $\sqrt{(-5)^2} = \sqrt{25} = 5 = |-5|$. To avoid writing absolute values constantly, in the rest of this chapter we assume that variables under even roots are positive unless stated otherwise.

11.2 SIMPLIFYING RADICALS

Theorem 11.3 (Product and quotient rules). If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ and, for $b \neq 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

A radical is in *simplest form* when the radicand has no factor that is a perfect n th power, there is no fraction under the radical, and there is no radical in a denominator. To simplify, factor out the largest perfect n th power.

Example 11.4 (Simplifying radicals). (a) $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$. (b) $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = 3\sqrt[3]{2}$. (c) $\sqrt{18x^5y^2} = \sqrt{9x^4y^2 \cdot 2x} = 3x^2y\sqrt{2x}$. (d) $\sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$.

COMMON ERROR (Roots do not distribute over sums). $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. For instance $\sqrt{9+16} = \sqrt{25} = 5$, while $\sqrt{9} + \sqrt{16} = 7$. The product rule is about products only.

11.3 OPERATIONS WITH RADICALS

Like radicals have the same index and the same radicand, and they combine as like terms do: $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$. Radicals that look unlike may become like after simplifying.

Example 11.5 (Adding radicals). $3\sqrt{12} - \sqrt{27} + \sqrt{3} = 3 \cdot 2\sqrt{3} - 3\sqrt{3} + \sqrt{3} = 4\sqrt{3}$.

Radicals multiply by the product rule and the distributive property, including the special products.

Example 11.6 (Multiplying radicals). (a) $(2 + \sqrt{3})(1 - \sqrt{3}) = 2 - 2\sqrt{3} + \sqrt{3} - 3 = -1 - \sqrt{3}$. (b) $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) = 5 - 2 = 3$.

Example (b) shows why *conjugates* $a + \sqrt{b}$ and $a - \sqrt{b}$ are useful: their product has no radical. To *rationalize* a denominator, multiply the numerator and denominator by whatever removes the radical from the denominator: the radical itself for a single term, the conjugate for a binomial.

Example 11.7 (Rationalizing denominators). (a) $\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$.

$$(b) \frac{4}{3 - \sqrt{5}} = \frac{4(3 + \sqrt{5})}{(3 - \sqrt{5})(3 + \sqrt{5})} = \frac{4(3 + \sqrt{5})}{9 - 5} = 3 + \sqrt{5}.$$

11.4 RATIONAL EXPONENTS

We want the laws of exponents to hold for fractional exponents too. If they do, then $(a^{1/n})^n = a^{n/n} = a$, so $a^{1/n}$ must be an n th root of a . This forces the definition.

Definition 11.8 (Rational exponents). For a positive integer n and an integer m , whenever $\sqrt[n]{a}$ is real,

$$a^{1/n} = \sqrt[n]{a} \quad \text{and} \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

With this definition, all the laws of exponents in Theorem 2.8 hold for rational exponents (with positive bases). It is usually easiest to take the root first, since it keeps the numbers small.

Example 11.9 (Rational exponents). (a) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$. (b)

$$16^{-3/4} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{8}. \quad (c) (x^{1/2}y^{-1/3})^6 = x^3y^{-2} = \frac{x^3}{y^2}. \quad (d) \sqrt{x} \cdot \sqrt[3]{x} = x^{1/2}x^{1/3} = x^{5/6} = \sqrt[6]{x^5}.$$

Part (d) shows the main advantage of exponent notation: radicals with different

indices can be combined using the laws of exponents.

11.5 RADICAL EQUATIONS

To solve an equation containing a square root, isolate the radical and square both sides. Squaring is not reversible, though: $a = b$ implies $a^2 = b^2$, but $a^2 = b^2$ only implies $a = \pm b$. So squaring can introduce extraneous solutions, and every candidate must be checked in the *original* equation.

Example 11.10 (An extraneous solution). Solve $\sqrt{x+3} = x-3$.

Solution. Square both sides: $x+3 = x^2 - 6x + 9$, so $x^2 - 7x + 6 = 0$ and $(x-1)(x-6) = 0$. Check: $x = 6$ gives $\sqrt{9} = 3$, true. $x = 1$ gives $\sqrt{4} = 2$ on the left and -2 on the right, false. The only solution is $x = 6$. Figure 11.1 shows what happened: $x = 1$ is where the line meets the *negative* square root, which squaring cannot tell apart from the positive one.

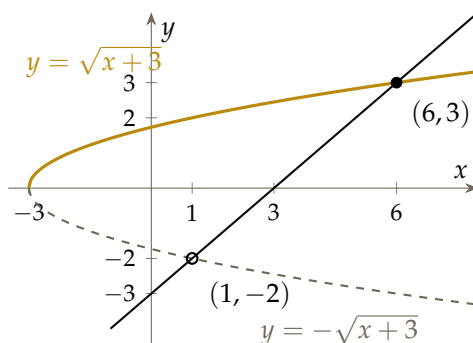


Figure 11.1: The line $y = x - 3$ meets $y = \sqrt{x+3}$ only at $(6, 3)$. Its other intersection, $(1, -2)$, is with the dashed curve $y = -\sqrt{x+3}$; squaring both sides merges the two curves, which is where the extraneous solution comes from.

Example 11.11 (Two radicals). Solve $\sqrt{x+5} - \sqrt{x} = 1$.

Solution. Isolate one radical: $\sqrt{x+5} = 1 + \sqrt{x}$. Square: $x+5 = 1 + 2\sqrt{x} + x$, so $2\sqrt{x} = 4$ and $\sqrt{x} = 2$. Then $x = 4$, which checks: $3 - 2 = 1$.

COMMON ERROR (Square sides, not terms). Squaring $\sqrt{x+5} = 1 + \sqrt{x}$ gives $(1 + \sqrt{x})^2 = 1 + 2\sqrt{x} + x$ on the right, not $1 + x$. The middle term is easy to forget.

11.6 COMPLEX NUMBERS

The equation $x^2 = -1$ has no real solution. We create one by defining a new number i with $i^2 = -1$, and then build a number system around it.

Definition 11.12 (Complex numbers). The *imaginary unit* i satisfies $i^2 = -1$. A *complex number* is a number $a + bi$ with a and b real; a is its *real part* and b its *imaginary part*. For $c > 0$, $\sqrt{-c} = i\sqrt{c}$. The *conjugate* of $a + bi$ is $a - bi$.

Complex numbers are added and multiplied like polynomials in i , replacing i^2 by -1 . To divide, multiply the numerator and denominator by the conjugate of the denominator, since $(a + bi)(a - bi) = a^2 + b^2$ is real.

Example 11.13 (Complex arithmetic). (a) $(3 + 2i) + (1 - 5i) = 4 - 3i$. (b) $(2 + 3i)(4 - i) = 8 - 2i + 12i - 3i^2 = 11 + 10i$.

$$(c) \frac{5 + i}{2 - 3i} = \frac{(5 + i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{10 + 17i + 3i^2}{13} = \frac{7 + 17i}{13} = \frac{7}{13} + \frac{17}{13}i.$$

The powers of i repeat in a cycle of four: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, then $i^5 = i$ again. So i^n depends only on the remainder when n is divided by 4; for example $i^{23} = i^{20}i^3 = -i$.

With complex numbers, the quadratic formula works even when the discriminant is negative.

Example 11.14 (Complex solutions). Solve $x^2 - 4x + 13 = 0$.

Solution. $x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$. Check $2 + 3i$: $(2 + 3i)^2 = 4 + 12i - 9 = -5 + 12i$, and $-5 + 12i - 4(2 + 3i) + 13 = 0$.

COMMON ERROR ($\sqrt{a}\sqrt{b} = \sqrt{ab}$ fails for negatives). $\sqrt{-4} \cdot \sqrt{-9} = (2i)(3i) = 6i^2 = -6$, not $\sqrt{36} = 6$. Always rewrite $\sqrt{-c}$ as $i\sqrt{c}$ before multiplying.

LOOKING AHEAD. Rational exponents are extended to all real exponents in Chapter 12, where 2^x makes sense for irrational x such as $\sqrt{2}$. Volume II develops the complex numbers geometrically, as points in a plane, and shows that every polynomial equation has a solution among them (the Fundamen-

tal Theorem of Algebra). In Volume IV, complex numbers are the language of quantum mechanics.

11.7 WORKED PROBLEMS

These problems practice radicals, rational exponents, and complex numbers. Simplify each radical before combining, and check every solution of a radical equation.

PROBLEM 11.1 Simplify before dividing

Simplify $\frac{\sqrt{12} + \sqrt{27}}{\sqrt{3}}$.

Solution. $\sqrt{12} = 2\sqrt{3}$ and $\sqrt{27} = 3\sqrt{3}$, so the numerator is $5\sqrt{3}$ and the quotient is 5. Alternatively, divide each term: $\sqrt{\frac{12}{3}} + \sqrt{\frac{27}{3}} = 2 + 3 = 5$.

PROBLEM 11.2 Rationalizing with a variable

Rationalize the denominator of $\frac{\sqrt{x} - 2}{\sqrt{x} + 2}$.

Solution. Multiply the top and bottom by the conjugate $\sqrt{x} - 2$:

$$\frac{(\sqrt{x} - 2)^2}{(\sqrt{x})^2 - 4} = \frac{x - 4\sqrt{x} + 4}{x - 4}, \quad x \neq 4.$$

PROBLEM 11.3 A rational exponent of a product

Simplify $(27x^6y^{-3})^{2/3}$.

Solution. Apply the exponent to each factor: $27^{2/3} = (\sqrt[3]{27})^2 = 9$, $(x^6)^{2/3} = x^4$, and $(y^{-3})^{2/3} = y^{-2}$. The result is $\frac{9x^4}{y^2}$.

PROBLEM 11.4 Two radicals, two solutions

Solve $\sqrt{2x+3} - \sqrt{x+1} = 1$.

Solution. Isolate one radical: $\sqrt{2x+3} = 1 + \sqrt{x+1}$. Squaring: $2x+3 = 1 + 2\sqrt{x+1} + x+1$, so $x+1 = 2\sqrt{x+1}$. Let $u = \sqrt{x+1}$; then $u^2 = 2u$, so

$u = 0$ or $u = 2$, giving $x = -1$ or $x = 3$. Check: $x = -1$ gives $\sqrt{1} - \sqrt{0} = 1$, and $x = 3$ gives $\sqrt{9} - \sqrt{4} = 1$. Both are solutions; squaring does not always create extraneous ones, but only checking tells.

PROBLEM 11.5 A complex quotient

Write $\frac{4-2i}{1+3i}$ in the form $a+bi$, and check the answer.

Solution. Multiply by the conjugate of the denominator:

$$\frac{(4-2i)(1-3i)}{(1+3i)(1-3i)} = \frac{4-12i-2i+6i^2}{1+9} = \frac{-2-14i}{10} = -\frac{1}{5} - \frac{7}{5}i.$$
 Check:
 $(1+3i)(-\frac{1}{5} - \frac{7}{5}i) = -\frac{1}{5} - \frac{7}{5}i - \frac{3}{5}i - \frac{21}{5}i^2 = 4-2i.$

PROBLEM 11.6 A distance with radicals

Find the distance between $(1+\sqrt{3}, 2)$ and $(1-\sqrt{3}, 0)$.

Solution. The differences are $2\sqrt{3}$ and 2, so $d = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4$. Squaring $2\sqrt{3}$ gives $4 \cdot 3 = 12$, not $2 \cdot 3$.

PROBLEM 11.7 Kepler's third law

A planet's orbital period T (in years) and average distance a from the Sun (in astronomical units) satisfy $T = a^{3/2}$. Find Mars's period ($a = 1.52$), and the distance of a planet with an 8-year period.

Solution. For Mars, $T = 1.52^{3/2} = 1.52\sqrt{1.52} \approx 1.52(1.233) \approx 1.87$ years. For $T = 8$, solve $a^{3/2} = 8$ by raising both sides to the power $\frac{2}{3}$: $a = 8^{2/3} = (\sqrt[3]{8})^2 = 4$ AU.

PROBLEM 11.8 Complex roots and their sum and product

Solve $x^2 + 2x + 5 = 0$, and check that the sum and product of the roots are $-\frac{b}{a}$ and $\frac{c}{a}$.

Solution. $x = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$. Sum: $(-1+2i) + (-1-2i) = -2 = -\frac{2}{1}$. Product: $(-1)^2 - (2i)^2 = 1+4 = 5 = \frac{5}{1}$. The roots are conjugates, as they always are for a real quadratic with negative discriminant.

SKILL DRILLS

Short problems for building speed and accuracy. Variables are positive. Answers follow the list.

D11.1 $\sqrt{64}$

D11.2 $\sqrt[3]{-125}$

D11.3 $\sqrt{-25}$

D11.4 $\sqrt{45}$

D11.5 $\sqrt{98}$

D11.6 $\sqrt[3]{24}$

D11.7 $2\sqrt{3} + 5\sqrt{3}$

D11.8 $\sqrt{8} + \sqrt{2}$

D11.9 $\sqrt{2} \cdot \sqrt{8}$

D11.10 $(\sqrt{5})^2$

D11.11 Rationalize $\frac{1}{\sqrt{2}}$

D11.12 Rationalize $\frac{2}{\sqrt{3}-1}$

D11.13 $9^{1/2}$

D11.14 $8^{1/3}$

D11.15 $4^{3/2}$

D11.16 $27^{-1/3}$

D11.17 $x^{1/2} \cdot x^{1/2}$

D11.18 $(x^{1/3})^6$

D11.19 Solve $\sqrt{x} = 7$

D11.20 Solve $\sqrt{x+1} = -2$

D11.21 i^2

D11.22 $(1+i)^2$

D11.23 $(2+i)(2-i)$

D11.24 Solve $x^2 = -9$

ANSWERS TO SKILL DRILLS

D11.1 8. D11.2 -5 . D11.3 $5i$. D11.4 $3\sqrt{5}$. D11.5 $7\sqrt{2}$. D11.6 $2\sqrt[3]{3}$. D11.7 $7\sqrt{3}$. D11.8 $3\sqrt{2}$. D11.9 4. D11.10 5. D11.11 $\frac{\sqrt{2}}{2}$. D11.12 $\sqrt{3} + 1$. D11.13 3. D11.14 2. D11.15 8. D11.16 $\frac{1}{3}$. D11.17 x . D11.18 x^2 . D11.19 $x = 49$. D11.20 No solution. D11.21 -1 . D11.22 $2i$. D11.23 5. D11.24 $x = \pm 3i$.

EXERCISES

COMPUTATION

11.1 Evaluate $\sqrt{121}$, $\sqrt[3]{-64}$, $\sqrt[4]{81}$ and $\sqrt[5]{-32}$.

11.2 Which are real numbers: $\sqrt{-9}$, $\sqrt[3]{-8}$, $\sqrt[4]{-1}$?

11.3 For every real x , simplify $\sqrt{x^2}$ and $\sqrt[3]{x^3}$.

11.4 Simplify $\sqrt{75}$.

- 11.5** Simplify $\sqrt{200}$.
- 11.6** Simplify $\sqrt[3]{40}$.
- 11.7** Simplify $\sqrt{48a^3b^4}$.
- 11.8** Simplify $\sqrt[3]{16x^7}$.
- 11.9** Simplify $\sqrt{\frac{7}{16}}$.
- 11.10** Simplify $2\sqrt{18} + 3\sqrt{8}$.
- 11.11** Simplify $\sqrt{50} - \sqrt{32} + \sqrt{2}$.
- 11.12** Simplify $4\sqrt[3]{16} - \sqrt[3]{54}$.
- 11.13** Simplify $\sqrt{6} \cdot \sqrt{15}$.
- 11.14** Multiply: $\sqrt{3}(\sqrt{12} - 2)$.
- 11.15** Multiply: $(3 + \sqrt{2})(4 - \sqrt{2})$.
- 11.16** Multiply: $(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})$.
- 11.17** Expand $(2\sqrt{5} - 1)^2$.
- 11.18** Rationalize the denominator: $\frac{10}{\sqrt{5}}$.
- 11.19** Rationalize the denominator: $\frac{\sqrt{2}}{\sqrt{6}}$.
- 11.20** Rationalize the denominator: $\frac{3}{2 + \sqrt{3}}$.
- 11.21** Rationalize the denominator: $\frac{\sqrt{5} + 1}{\sqrt{5} - 1}$.
- 11.22** Evaluate $25^{1/2}$, $27^{2/3}$, $32^{-2/5}$ and $(\frac{1}{16})^{-3/4}$.
- 11.23** Write $x^{3/4}$ as a radical, and $\sqrt[3]{y^2}$ with a rational exponent.
- 11.24** Simplify $x^{2/3} \cdot x^{1/6}$.
- 11.25** Simplify $(a^{3/4})^8$.
- 11.26** Simplify $(16x^8y^{-4})^{1/4}$.
- 11.27** Write $\sqrt{x} \cdot \sqrt[4]{x}$ as a single radical.

- 11.28** Solve $\sqrt{x-4} = 3$.
- 11.29** Solve $\sqrt{3x+1} = x-3$.
- 11.30** Solve $\sqrt{x+7} = x+1$.
- 11.31** Solve $\sqrt[3]{2x-1} = 3$.
- 11.32** Solve $\sqrt{x+1} + \sqrt{x-4} = 5$.
- 11.33** Solve $x^{2/3} = 16$.
- 11.34** The period of a pendulum of length L meters is $T = 2\pi\sqrt{\frac{L}{9.8}}$ seconds. What length gives a period of 2 seconds?
- 11.35** Simplify i^7 , i^{20} and i^{-1} .
- 11.36** Write $\sqrt{-49}$ and $\sqrt{-12}$ in terms of i .
- 11.37** Simplify $(4-3i) + (-1+6i)$ and $(2+i) - (5-4i)$.
- 11.38** Multiply: $(3+2i)(1-4i)$.
- 11.39** Expand $(2-5i)^2$.
- 11.40** Divide: $\frac{1+i}{1-i}$.
- 11.41** Divide: $\frac{3+4i}{2+i}$.
- 11.42** Solve $x^2 + 25 = 0$.
- 11.43** Solve $x^2 - 6x + 10 = 0$.
- 11.44** Solve $2x^2 + 2x + 5 = 0$.

CONCEPTS AND PROOF

- 11.45** Explain why $\sqrt{x^2} = |x|$ rather than x . Give a value of x for which $\sqrt{x^2} \neq x$.
- 11.46** Show that $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ in general, using $a = 9$, $b = 16$.
- 11.47** Show that $\sqrt{-4} \cdot \sqrt{-9} \neq \sqrt{(-4)(-9)}$, and explain which rule fails.
- 11.48** Why must the solutions of a radical equation be checked? Relate your answer to the fact that $(-2)^2 = 2^2$.
- 11.49** Prove that $(a+bi)(a-bi) = a^2 + b^2$, so the product of a complex number and its conjugate is a nonnegative real number.

11.50 Use the law $(a^m)^n = a^{mn}$ to explain why $a^{1/2}$ must equal $\sqrt{a}$ for $a \geq 0$.

CHALLENGE

11.51 Simplify $\sqrt{7 + 4\sqrt{3}}$. *Hint:* look for $(p + q\sqrt{3})^2$.

11.52 Solve $x - 3\sqrt{x} - 4 = 0$.

11.53 Find all complex numbers z with $z^2 = i$. *Hint:* write $z = a + bi$ and compare real and imaginary parts.

CHAPTER 12

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LEARNING OBJECTIVES

- Evaluate and graph exponential functions, and distinguish growth from decay.
- Model compound interest, population growth, and radioactive decay.
- Define logarithms as exponents, evaluate them, and convert between logarithmic and exponential form.
- State, prove, and apply the properties of logarithms, including the change of base formula.
- Solve exponential and logarithmic equations.

In every function so far, the variable has been the base: x^2 , x^3 , $\sqrt{x}$. Putting the variable in the *exponent* produces a new kind of function, one that models anything growing or shrinking by a fixed percentage per unit of time: money, populations, radioactive material, the spread of information. The inverse of such a function, the logarithm, turns multiplication into addition and lets us solve for an unknown exponent.

12.1 EXPONENTIAL FUNCTIONS

Definition 12.1 (Exponential function). For a constant $b > 0$ with $b \neq 1$, the *exponential function with base b* is $f(x) = b^x$. Its domain is all real numbers and its range is $(0, \infty)$.

For rational x , b^x was defined in Chapter 11; for irrational x , such as $2^{\sqrt{2}}$, it is

defined by approximating x with rationals ($2^{1.4}, 2^{1.41}, 2^{1.414}, \dots$), and the results approach a single number. The base must be positive (otherwise $(-4)^{1/2}$ is not real) and not 1 (since $1^x = 1$ is constant).

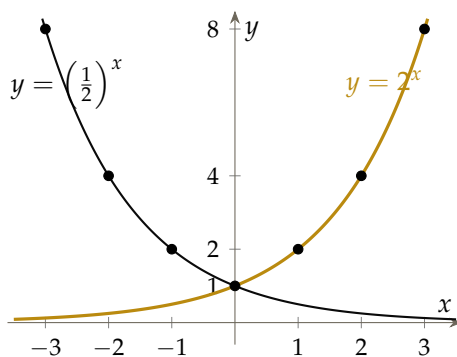


Figure 12.1: Exponential growth ($b = 2$) and decay ($b = \frac{1}{2}$). Both graphs pass through $(0, 1)$, stay above the x -axis, and approach it on one side without touching.

If $b > 1$, b^x increases: this is *exponential growth*. If $0 < b < 1$, b^x decreases toward 0: *exponential decay*. Since $(\frac{1}{2})^x = 2^{-x}$, the graph of $(\frac{1}{2})^x$ is the reflection of 2^x across the y -axis (Fig. 12.1). The x -axis is a *horizontal asymptote*: the graph gets arbitrarily close to it but never reaches it.

COMMON ERROR (2^x is not x^2). In x^2 the base varies; in 2^x the exponent varies. They agree at $x = 2$ and $x = 4$, but 2^x eventually grows far faster: 2^{20} is over a million, while $20^2 = 400$.

12.2 GROWTH AND DECAY

A quantity that changes by the same *percentage* in each period of time is exponential. If it starts at A_0 and is multiplied by b each period, after t periods it is $A(t) = A_0 b^t$.

Example 12.2 (Doubling). A culture of 500 bacteria doubles every 3 hours. Find a formula for the population, and the population after 12 hours.

Solution. In t hours there are $\frac{t}{3}$ doublings, so $N(t) = 500 \cdot 2^{t/3}$. After 12 hours, $N = 500 \cdot 2^4 = 8000$.

Theorem 12.3 (Compound interest). A principal P invested at annual rate r , compounded n times per year, grows in t years to

$$A = P \left(1 + \frac{r}{n} \right)^{nt}.$$

Compounded continuously, it grows to $A = Pe^{rt}$, where $e \approx 2.71828$.

Each compounding period multiplies the balance by $1 + \frac{r}{n}$, and there are nt periods. As n increases, $\left(1 + \frac{r}{n}\right)^{nt}$ approaches e^{rt} ; the number e is the limit of $\left(1 + \frac{1}{n}\right)^n$ as n grows without bound, and it is the most important base in mathematics.

Example 12.4 (Comparing compounding). \$5000 is invested at 4% for 10 years. Compounded monthly, it grows to $5000 \left(1 + \frac{0.04}{12}\right)^{120} \approx \$7454.16$; compounded continuously, to $5000e^{0.4} \approx \$7459.12$. Continuous compounding earns only a little more.

Example 12.5 (Radioactive decay). Carbon-14 has a *half-life* of 5730 years: half of any sample decays in that time. How much of a 100 g sample remains after 2000 years?

Solution. $A(t) = 100 \left(\frac{1}{2}\right)^{t/5730}$, so $A(2000) = 100 \left(\frac{1}{2}\right)^{2000/5730} \approx 78.51$ g.

12.3 LOGARITHMS

To answer “when does the population reach 10,000?” we must solve for an exponent. Logarithms are the tool.

Definition 12.6 (Logarithm). Let $b > 0$, $b \neq 1$. For $x > 0$, the *logarithm of x with base b* is the exponent to which b must be raised to give x :

$$\log_b x = y \quad \text{means} \quad b^y = x.$$

The *common logarithm* is $\log x = \log_{10} x$, and the *natural logarithm* is $\ln x = \log_e x$.

A logarithm *is* an exponent. To evaluate $\log_b x$, ask: “ b to what power gives x ?”

Example 12.7 (Evaluating logarithms). $\log_2 32 = 5$ because $2^5 = 32$. $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$. $\log_9 3 = \frac{1}{2}$ because $9^{1/2} = 3$. $\log_5 1 = 0$ because $5^0 = 1$. And $\ln e^4 = 4$.

Theorem 12.8 (Basic properties). $\log_b 1 = 0$, $\log_b b = 1$, $\log_b b^x = x$ for every real x , and $b^{\log_b x} = x$ for every $x > 0$.

The last two properties say that $f(x) = b^x$ and $g(x) = \log_b x$ undo each other: they are *inverse functions*. So the graph of $y = \log_b x$ is the reflection of $y = b^x$ across the line $y = x$ (Fig. 12.2). Its domain is $(0, \infty)$, so the logarithm of zero or a negative number is undefined, and the y -axis is a vertical asymptote.

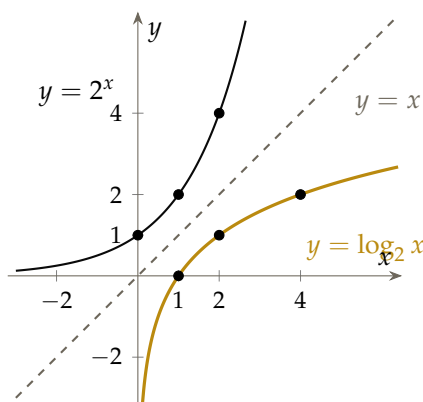


Figure 12.2: $y = \log_2 x$ is the reflection of $y = 2^x$ in the line $y = x$: the point $(2, 4)$ on one corresponds to $(4, 2)$ on the other.

12.4 PROPERTIES OF LOGARITHMS

Because logarithms are exponents, the laws of exponents become laws of logarithms.

Theorem 12.9 (Laws of logarithms). For $x, y > 0$ and any real r :

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b \frac{x}{y} = \log_b x - \log_b y,$$

$$\log_b(x^r) = r \log_b x.$$

Proof of the product law. Let $m = \log_b x$ and $n = \log_b y$, so $x = b^m$ and $y = b^n$.

Then $xy = b^m b^n = b^{m+n}$, and by definition $\log_b(xy) = m + n = \log_b x + \log_b y$. The other laws follow in the same way from $\frac{b^m}{b^n} = b^{m-n}$ and $(b^m)^r = b^{mr}$. ■

Example 12.10 (Expanding and combining). (a) $\log_3 \frac{9x^2}{y} = \log_3 9 + \log_3 x^2 - \log_3 y = 2 + 2\log_3 x - \log_3 y$.
(b) $2\ln x + \ln 3 - \ln(x+1) = \ln x^2 + \ln 3 - \ln(x+1) = \ln \frac{3x^2}{x+1}$.

COMMON ERROR (What the laws do *not* say). $\log_b(x+y) \neq \log_b x + \log_b y$, and $\frac{\log_b x}{\log_b y} \neq \log_b \frac{x}{y}$. The laws turn *products* into sums, and quotients into differences, never the other way. Test: $\log(1+1) = \log 2 \approx 0.301$, but $\log 1 + \log 1 = 0$.

Calculators have keys for log and ln only. Other bases are handled by the following formula, proved in Exercise 43.

Theorem 12.11 (Change of base). $\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}$.

For example, $\log_2 10 = \frac{\ln 10}{\ln 2} \approx 3.3219$; and indeed $2^{3.3219} \approx 10$.

12.5 EXPONENTIAL AND LOGARITHMIC EQUATIONS

To solve an equation with the variable in an exponent, isolate the exponential and take a logarithm of both sides. If both sides can be written as powers of the same base, the exponents can simply be equated, since b^x is one-to-one.

Example 12.12 (Exponential equations). (a) $2^{x+1} = 32 = 2^5$, so $x+1 = 5$ and $x = 4$. (b) $3^x = 20$: take ln of both sides, $x \ln 3 = \ln 20$, so $x = \frac{\ln 20}{\ln 3} \approx 2.727$.

Example 12.13 (Doubling time). How long does it take an investment to double at 4% compounded annually?

Solution. Solve $P(1.04)^t = 2P$, that is, $(1.04)^t = 2$. Taking ln: $t \ln 1.04 = \ln 2$, so $t = \frac{\ln 2}{\ln 1.04} \approx 17.67$ years. The principal P cancels: the doubling time

does not depend on the amount invested.

To solve an equation containing logarithms, combine them into a single logarithm and rewrite in exponential form. Since logarithms are defined only for positive numbers, check every solution in the original equation.

Example 12.14 (A logarithmic equation). Solve $\log x + \log(x - 3) = 1$.

Solution. Combine: $\log(x(x - 3)) = 1$, so $x(x - 3) = 10^1$, $x^2 - 3x - 10 = 0$, and $(x - 5)(x + 2) = 0$. The candidate $x = -2$ makes $\log x$ undefined, so it is extraneous. The solution is $x = 5$: $\log 5 + \log 2 = \log 10 = 1$.

LOOKING AHEAD. Volume II develops these functions in depth: logistic growth, Newton's law of cooling, and using logarithms to find exponential patterns in data. In Volume III, e^x turns out to be the unique function equal to its own derivative, which is why e , not 10, is the natural base, and $\ln x$ appears as the integral of $\frac{1}{x}$.

12.6 WORKED PROBLEMS

These problems model growth and decay and solve for unknown exponents. Isolate the exponential before taking logarithms.

PROBLEM 12.1 Unknowns in two exponents

Solve $2^{3x} = 5^{x-1}$.

Solution. Take $\ln$ of both sides and use the power law: $3x \ln 2 = (x - 1) \ln 5$. Collect the x -terms: $x(3 \ln 2 - \ln 5) = -\ln 5$, so

$$x = \frac{\ln 5}{\ln 5 - 3 \ln 2} = \frac{\ln 5}{\ln \frac{5}{8}} \approx -3.424.$$

PROBLEM 12.2 A growth model from data

A town had 20,000 people in 2015 and 26,000 in 2020. Assuming exponential growth, predict its population in 2030, and find when it reaches 40,000.

Solution. With t years after 2015, $P = 20,000 b^t$, and $b^5 = \frac{26,000}{20,000} = 1.3$. So $P = 20,000(1.3)^{t/5}$; the annual growth factor is $1.3^{1/5} \approx 1.054$, about 5.4%

per year. In 2030, $t = 15$ and $P = 20,000(1.3)^3 = 43,940$. For $P = 40,000$: $(1.3)^{t/5} = 2$, so $t = \frac{5 \ln 2}{\ln 1.3} \approx 13.2$ years, during 2028.

PROBLEM 12.3 Radiocarbon dating

A bone contains 30% of the carbon-14 it had when the animal died. The half-life of carbon-14 is 5730 years. How old is the bone?

Solution. $(\frac{1}{2})^{t/5730} = 0.3$. Taking $\ln$: $\frac{t}{5730} \ln \frac{1}{2} = \ln 0.3$, so $t = \frac{5730 \ln 0.3}{\ln 0.5} \approx 9953$ years. Both logarithms are negative, so t is positive, as it must be.

PROBLEM 12.4 A logarithmic equation

Solve $\ln x + \ln(x + 2) = \ln 15$.

Solution. $\ln(x(x + 2)) = \ln 15$, and since $\ln$ is one-to-one, $x^2 + 2x = 15$, so $(x + 5)(x - 3) = 0$. The candidate $x = -5$ is not in the domain of $\ln x$, so the only solution is $x = 3$.

PROBLEM 12.5 Two bases

Solve $\log_2 x + \log_4 x = 6$.

Solution. Change $\log_4 x$ to base 2: $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{1}{2} \log_2 x$. So $\frac{3}{2} \log_2 x = 6$, $\log_2 x = 4$ and $x = 16$. *Check:* $4 + 2 = 6$.

PROBLEM 12.6 Comparing two accounts

Which is better: 6% compounded annually or 5.85% compounded continuously?

Solution. Compare the growth in one year. Annual compounding multiplies by 1.06; continuous compounding by $e^{0.0585} \approx 1.06024$. The continuous account is slightly better, earning an effective rate of about 6.02%.

PROBLEM 12.7 An exponential equation quadratic in form

Solve $4^x - 3 \cdot 2^x - 4 = 0$.

Solution. Since $4^x = (2^x)^2$, let $u = 2^x$: $u^2 - 3u - 4 = (u - 4)(u + 1) = 0$. Then $2^x = 4$ gives $x = 2$, while $2^x = -1$ is impossible because $2^x > 0$. The only solution is $x = 2$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|-----------------------------------|---|
| D12.1 2^5 | D12.14 $\log_3 18 - \log_3 2$ |
| D12.2 3^{-2} | D12.15 $\log_2 (8^5)$ |
| D12.3 $(\frac{1}{2})^{-3}$ | D12.16 $\ln(e^3 e^2)$ |
| D12.4 e^0 | D12.17 Solve $3^x = 81$ |
| D12.5 $\log_2 8$ | D12.18 Solve $10^x = 0.01$ |
| D12.6 $\log_3 \frac{1}{9}$ | D12.19 Solve $2^{x-1} = 16$ |
| D12.7 $\log 10,000$ | D12.20 Solve $\log_2 x = 5$ |
| D12.8 $\ln e$ | D12.21 Solve $\log x = -1$ |
| D12.9 $\log_5 1$ | D12.22 Solve $\ln x = 0$ |
| D12.10 $\log_8 2$ | D12.23 Domain of $\ln(x + 3)$ |
| D12.11 $10^{\log 7}$ | D12.24 Fraction left after 30 years if the half-life is 10 years |
| D12.12 $\log_4 64$ | |
| D12.13 $\log 2 + \log 5$ | |

ANSWERS TO SKILL DRILLS

D12.1 32. D12.2 $\frac{1}{9}$. D12.3 8. D12.4 1. D12.5 3. D12.6 -2 . D12.7 4. D12.8 1. D12.9 0. D12.10 $\frac{1}{3}$. D12.11 7. D12.12 3. D12.13 1. D12.14 2. D12.15 15. D12.16 5. D12.17 $x = 4$. D12.18 $x = -2$. D12.19 $x = 5$. D12.20 $x = 32$. D12.21 $x = 0.1$. D12.22 $x = 1$. D12.23 $x > -3$. D12.24 $\frac{1}{8}$.

EXERCISES

COMPUTATION

- 12.1** Evaluate $f(x) = 3^x$ at $x = -2, 0, 2$ and $\frac{1}{2}$.
- 12.2** Graph $y = 2^x$ and $y = 2^{-x}$ on the same axes. What do they have in common?
- 12.3** Which represent growth and which decay: 1.05^x , 0.8^x , $(\frac{3}{2})^{-x}$, e^x ?
- 12.4** Find the value of \$8000 invested at 5% compounded quarterly for 6 years.

- 12.5** Find the value of \$8000 invested at 5% compounded continuously for 6 years.
- 12.6** A \$30,000 car loses 15% of its value each year. Find its value after 5 years.
- 12.7** A population of 200 bacteria triples every 4 hours. How many are there after 12 hours?
- 12.8** Iodine-131 has a half-life of 8 days. How much of a 50 mg sample remains after 24 days?
- 12.9** Write in logarithmic form: $2^6 = 64$; $10^{-2} = 0.01$; $e^0 = 1$.
- 12.10** Write in exponential form: $\log_5 125 = 3$; $\log_4 2 = \frac{1}{2}$.
- 12.11** Evaluate $\log_3 81$, $\log_2 \frac{1}{8}$, $\log 1000$ and $\log_4 8$.
- 12.12** Evaluate $\ln e^7$, $e^{\ln 5}$, $\log_7 7$ and $\log_6 1$.
- 12.13** Find the domains of $\log_2(x - 5)$ and $\ln(4 - x^2)$.
- 12.14** Expand $\log_2(8x^3)$.
- 12.15** Expand $\ln \frac{x^2\sqrt{y}}{z}$.
- 12.16** Expand $\log \frac{x+1}{10x}$.
- 12.17** Write as a single logarithm: $\log_3 x + 2\log_3 y$.
- 12.18** Write as a single logarithm: $\ln(x^2 - 4) - \ln(x + 2)$.
- 12.19** Write as a single logarithm and evaluate: $3\log 2 + \log 5 - \log 4$.
- 12.20** Use the change of base formula to evaluate $\log_3 50$ and $\log_7 2$ to four decimal places.
- 12.21** Solve $2^x = 64$.
- 12.22** Solve $9^{x-1} = 27$.
- 12.23** Solve $5^x = 40$.
- 12.24** Solve $e^{2x} = 15$.
- 12.25** Solve $3 \cdot 2^x + 4 = 52$.
- 12.26** Solve $4^x = 3^{x+1}$.

- 12.27** Solve $\log_3 x = 4$.
- 12.28** Solve $\log(2x + 1) = 2$.
- 12.29** Solve $\ln(x - 1) = 3$.
- 12.30** Solve $\log_2 x + \log_2(x - 2) = 3$.
- 12.31** Solve $\log(x + 2) - \log x = 1$.
- 12.32** How long does it take \$1000 to double at 6% compounded annually?
- 12.33** How long does it take an investment to triple at 3% compounded continuously?
- 12.34** A substance decays from 80 g to 60 g in 10 days. Find its half-life.
- 12.35** The pH of a solution is $-\log[\text{H}^+]$. Find the pH when $[\text{H}^+] = 3.2 \times 10^{-4}$, and $[\text{H}^+]$ when the pH is 7.4.
- 12.36** The magnitude of an earthquake is $M = \log \frac{I}{I_0}$. How many times more intense is an earthquake of magnitude 7.2 than one of magnitude 5.2?

CONCEPTS AND PROOF

- 12.37** Explain why the base of an exponential function must be positive and different from 1.
- 12.38** Prove the product law $\log_b(xy) = \log_b x + \log_b y$.
- 12.39** Prove the power law $\log_b(x^r) = r \log_b x$.
- 12.40** Explain why $\log_b x$ is undefined for $x \leq 0$.
- 12.41** Show that $\log(a + b) \neq \log a + \log b$ in general.
- 12.42** Explain why the graphs of $y = b^x$ and $y = \log_b x$ are reflections of each other in the line $y = x$.
- 12.43** Prove the change of base formula. *Hint:* let $y = \log_b x$, so $b^y = x$, and take $\ln$ of both sides.

CHALLENGE

- 12.44** Solve $e^{2x} - 5e^x + 6 = 0$. *Hint:* let $u = e^x$.
- 12.45** Solve $\log_2(\log_3 x) = 1$.

- 12.46** Explain the “rule of 70”: a quantity growing at $p\%$ per year doubles in about $\frac{70}{p}$ years. Compare the exact doubling time at 7%.

APPENDIX A

FORMULA REFERENCE

This appendix collects the rules and formulas of the book for quick reference. The chapter where each is developed is given in parentheses.

A.1 NUMBERS AND EXPRESSIONS

Properties of the real numbers (Chapter 1): $a + b = b + a$, $ab = ba$, $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$, $a(b + c) = ab + ac$, $a + 0 = a$, $a \cdot 1 = a$, $a + (-a) = 0$, $a \cdot \frac{1}{a} = 1$ for $a \neq 0$.

Fractions (Chapter 1):

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Order of operations (Chapter 1): grouping; powers; multiplication and division left to right; addition and subtraction left to right.

Laws of exponents (Chapters 2 and 11): for $a, b > 0$ and rational m, n ,

$$a^m a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn}, \quad (ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$

$$a^0 = 1, \quad a^{-n} = \frac{1}{a^n}, \quad a^{1/n} = \sqrt[n]{a}, \quad a^{m/n} = (\sqrt[n]{a})^m.$$

A.2 EQUATIONS AND INEQUALITIES

Absolute value (Chapter 3): for $a > 0$, $|X| = a \iff X = \pm a$; $|X| < a \iff -a < X < a$; $|X| > a \iff X < -a$ or $X > a$.

Inequalities (Chapter 3): multiplying or dividing both sides by a negative number reverses the inequality.

Zero Product Property (Chapter 8): $ab = 0 \iff a = 0$ or $b = 0$.

Quadratic formula (Chapter 9): the solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant $b^2 - 4ac$ is positive (two real solutions), zero (one), or negative (no real solutions; two complex ones).

A.3 LINES AND GRAPHS

Distance and midpoint (Chapter 4): $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$; midpoint $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Lines (Chapter 4): slope $m = \frac{y_2 - y_1}{x_2 - x_1}$; point-slope $y - y_1 = m(x - x_1)$; slope-intercept $y = mx + b$; standard $Ax + By = C$. Parallel lines have equal slopes; perpendicular lines have $m_1 m_2 = -1$.

Average rate of change (Chapter 5): $\frac{f(b) - f(a)}{b - a}$.

Parabolas (Chapter 9): $f(x) = a(x - h)^2 + k$ has vertex (h, k) ; for $f(x) = ax^2 + bx + c$, the vertex is at $x = -\frac{b}{2a}$.

A.4 POLYNOMIALS AND FACTORING

Special products and factoring patterns (Chapters 7 and 8):

$$\begin{aligned} (a + b)^2 &= a^2 + 2ab + b^2, & (a - b)^2 &= a^2 - 2ab + b^2, \\ a^2 - b^2 &= (a - b)(a + b), & (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\ a^3 - b^3 &= (a - b)(a^2 + ab + b^2), & a^3 + b^3 &= (a + b)(a^2 - ab + b^2). \end{aligned}$$

Division Algorithm (Chapter 7): $P(x) = D(x)Q(x) + R(x)$, with $R = 0$ or $\deg R < \deg D$.

A.5 RADICALS AND COMPLEX NUMBERS

Radicals (Chapter 11): $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ and $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$ when the roots are real; $\sqrt{x^2} = |x|$.

Complex numbers (Chapter 11): $i^2 = -1$; $\sqrt{-c} = i\sqrt{c}$ for $c > 0$; $(a + bi)(a - bi) = a^2 + b^2$.

A.6 EXPONENTIALS AND LOGARITHMS

Definition (Chapter 12): $\log_b x = y \iff b^y = x$, for $b > 0, b \neq 1, x > 0$.

Laws (Chapter 12):

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b \frac{x}{y} = \log_b x - \log_b y,$$

$$\log_b x^r = r \log_b x, \quad \log_b x = \frac{\ln x}{\ln b}.$$

Growth and decay (Chapter 12): $A = P \left(1 + \frac{r}{n}\right)^{nt}$; continuous $A = Pe^{rt}$; half-life h : $A = A_0 \left(\frac{1}{2}\right)^{t/h}$.

A.7 GEOMETRY

Rectangle	$A = lw, \quad P = 2l + 2w$
Triangle	$A = \frac{1}{2}bh$; angles sum to 180°
Right triangle	$a^2 + b^2 = c^2$ (Pythagorean Theorem)
Trapezoid	$A = \frac{1}{2}h(b_1 + b_2)$
Circle	$A = \pi r^2, \quad C = 2\pi r$
Rectangular box	$V = lwh, \quad S = 2(lw + lh + wh)$
Cylinder	$V = \pi r^2 h$
Sphere	$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$
Cone	$V = \frac{1}{3}\pi r^2 h$

ANSWERS TO SELECTED EXERCISES

Answers are given for odd-numbered exercises that have a short final answer. Every answer here was checked by computer algebra. Proofs are not answered; check a proof against the definitions and theorems it uses.

CHAPTER 1

NUMBERS AND ARITHMETIC

- 1.1** -5 : integer, rational. 0 : whole, integer, rational. $\frac{7}{2}$: rational. $\sqrt{16} = 4$: natural, whole, integer, rational. $\sqrt{7}$: irrational. $0.\overline{12} = \frac{4}{33}$: rational. $\frac{\pi}{2}$: irrational. 1.75 : rational.
- 1.3** $9; -4; 5; -3$
- 1.5** $-5; -15; 15; 0$
- 1.7** $-8; 7; 0$; no number q satisfies $0 \cdot q = -9$, so it is undefined.
- 1.9** $\frac{3}{4}; \frac{5}{7}; \frac{2}{5}$
- 1.11** (a) $\frac{17}{12}$; (b) $\frac{2}{15}$; (c) $\frac{1}{12}$
- 1.13** 5
- 1.15** $\frac{9}{20}; \frac{7}{9}; \frac{7}{11}; \frac{37}{30}$
- 1.17** -21
- 1.19** $\frac{13}{5}$
- 1.21** $594; 1530; 800$
- 1.23** (a) 36 ; (b) 25% ; (c) 75
- 1.25** $\$157.50; 37\%$
- 1.27** 85 miles
- 1.35** A net decrease of 25% , since $(1.5)(0.5) = 0.75$.
- 1.37** $\frac{1}{7}$
- 1.39** 648

CHAPTER 2

EXPRESSIONS AND EXPONENTS

- 2.1** 34
- 2.3** 2
- 2.5** Terms $4x^3, -x^2, \frac{2}{3}x, -9$; coefficients $4, -1, \frac{2}{3}$
- 2.7** 252
- 2.9** $2x^2 + 5x - 6$
- 2.11** $x^2 + 9x - 2$
- 2.13** $-4x - 5$
- 2.15** $x^8; y^{12}; 32a^5; m^5$
- 2.17** $-8a^9b^6$
- 2.19** $1; 1; -1; \frac{1}{8}; 9; \frac{1}{16}$
- 2.21** $\frac{x^6}{4y^8}$
- 2.23** $\frac{9m^8}{n^8}$
- 2.25** $9.3 \times 10^7; 7.1 \times 10^{-7}; 6.022 \times 10^{23}$
- 2.27** (a) 3×10^{-4} ; (b) 4×10^8
- 2.29** 500 s, about 8.3 minutes
- 2.33** No: $a = b = 1$ gives $4 \neq 2$. The square of side $a + b$ also contains two rectangles of area ab .
- 2.37** 1
- 2.39** 19 digits (4×10^{18})

CHAPTER 3

LINEAR EQUATIONS AND INEQUALITIES

- 3.1 $x = 4$
 3.3 $x = 7$
 3.5 $x = 6$
 3.7 $x = 28$
 3.9 $h = \frac{V}{\pi r^2}$
 3.11 $F = \frac{9}{5}C + 32$
 3.13 $x = \frac{b+d}{a-c}$
 3.15 46, 47, 48
 3.17 18 cm by 25 cm
 3.19 50 mph and 65 mph
 3.21 20 liters
 3.23 $(-\infty, 4)$
 3.25 $(3, \infty)$
 3.27 $[-1, 3)$
 3.29 $(-\infty, -2) \cup [3, \infty)$
 3.31 $x = -3$ or $x = 2$
 3.33 $(-7, 3)$
 3.35 $[-4, 8]$
 3.37 At least 86
 3.39 Dividing by x assumes $x \neq 0$; the solution $x = 0$ is lost. (The solutions are 0 and 5.)
 3.41 $|x - 6| < 4$
 3.43 Any negative x , such as $x = -1$. Multiplying by a negative x reverses the inequality.
 3.45 At $16\frac{4}{11}$ minutes after 3:00, about 3:16:22.

CHAPTER 4

GRAPHS AND LINES

- 4.1 IV; II; the positive y -axis; III
 4.3 13
 4.5 Two sides have length 5 and the third

$\sqrt{50}$, and $5^2 + 5^2 = 50$. Area 12.5.

- 4.7 $(-3, 0)$, $(3, 0)$, $(0, -9)$
 4.9 2
 4.11 Undefined; 0
 4.13 $y = 3x - 5$
 4.15 $y = -\frac{2}{3}x$
 4.17 $y = -7$; $x = 3$
 4.19 $y = 2x + 2$
 4.21 $y = -\frac{5}{2}x - 2$
 4.23 Opposite sides have slopes $\frac{1}{4}$ and 4; all four sides have length $\sqrt{17}$, so it is a rhombus.
 4.25 A line through $(3, 0)$ and $(0, 5)$
 4.27 $V = 24,000 - 2000t$; at $t = 12$ years
 4.29 12 years
 4.31 $k = -2$
 4.37 The triangle has a right angle at the origin exactly when $(1 + m_1^2) + (1 + m_2^2) = (m_1 - m_2)^2$, which simplifies to $m_1 m_2 = -1$.
 4.39 Dollars per item: the cost of each additional item.
 4.41 24

CHAPTER 5

FUNCTIONS

- 5.1 (a) Yes; (b) no; (c) yes; (d) no
 5.3 -5 ; -11 ; -4
 5.5 $-\frac{1}{2}$; 4; 0
 5.7 All real numbers
 5.9 $(-\infty, 5]$
 5.11 $[1, 5) \cup (5, \infty)$
 5.13 $f(-1) = 2.5$, $f(3) = -2.5$; $f(x) < 0$ on $[-3, -2)$ and $(1, 4)$
 5.15 $\frac{1}{5}$
 5.17 $f(x) = 3x + 1$
 5.19 Decreasing; -3 per unit

5.21 $C(w) = 6$ for $0 < w \leq 2$, $C(w) = 6 + 1.5(w - 2)$ for $w > 2$; \$6, \$10.50, \$18

5.23 Right 3, up 2; domain $[3, \infty)$, range $[2, \infty)$

5.25 Move $y = |x|$ down 5; reflect $y = x^2$ across the x -axis

5.27 x

5.29 $C(m) = 3.50 + 2.25m$; $C(8) = \$21.50$; about 11.8 miles

5.31 $x = 4$; $x > -\frac{1}{2}$

5.35 For $a < b$: $\frac{(mb+b_0)-(ma+b_0)}{b-a} = m$, writing b_0 for the intercept.

5.37 No real number squares to a negative number, but every real number has a real cube root.

5.39 -1 for $x < 0$ and 1 for $x > 0$; domain $x \neq 0$, range $\{-1, 1\}$

5.41 $\frac{(x+1)/(x-1)+1}{(x+1)/(x-1)-1} = \frac{2x}{2} = x$

CHAPTER 6

SYSTEMS OF LINEAR EQUATIONS

6.1 Yes

6.3 $(2, 4)$

6.5 $(2, -3)$

6.7 $(1, 2)$

6.9 $(-1, 2)$

6.11 $(2, 3)$

6.13 No solution

6.15 $(1, 2, 3)$

6.17 32 and 18

6.19 Boat 14 mph, current 2 mph

6.21 20 lb of \$10 coffee, 10 lb of \$16 coffee

6.23 80 units

6.25 $a = 3$, $b = 2$

6.27 Solid line $y = 2x - 3$, shaded above

6.29 $(0, 0)$, $(3, 3)$, $(0, 6)$

6.33 Equal quantities added to equal

quantities give equal sums, and the step can be reversed, so no solutions are gained or lost.

6.35 Infinitely many if $c = 8$; none otherwise

6.37 $x = \frac{1}{2}$, $y = \frac{1}{3}$

6.39 $\left(\frac{m-1}{m-2}, \frac{m^2-2}{m-2} \right)$

CHAPTER 7

POLYNOMIALS

7.1 $x^4 - 3x^2 - x + 7$; degree 4; leading coefficient 1

7.3 -17 ; 3

7.5 $4a^3 - 3a^2 + 5$

7.7 $-12x^5 + 6x^3 - 15x^2$

7.9 $6y^2 + 7y - 20$

7.11 $2x^3 - 7x^2 + 11x - 4$

7.13 $x^4 + x^2 + 1$

7.15 $16y^2 - 24y + 9$

7.17 $x^2 - 81$

7.19 $x^3 + 6x^2 + 12x + 8$

7.21 $x^2 + 2xy + y^2 - 9$

7.23 $3x^2 - 2x + 1$

7.25 $x + 4$

7.27 $3x^2 + x + 4$

7.29 $x^3 + 2x^2 + 4x + 8$

7.31 $x^2 + x + 1$

7.33 $8x + 16$

7.37 No: $x^3 + (-x^3 + 1) = 1$ has degree 0.

7.39 $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$

7.41 $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$

7.43 $k = \frac{10}{3}$

CHAPTER 8

FACTORING

8.1 $4x^2(2x - 3)$

8.3 $(x+7)(x-4)$

8.5 $(a+2)(b-3)$

8.7 $(x+4)(x+5)$

8.9 $(x+7)(x-4)$

8.11 $(x-3y)(x-5y)$

8.13 $(3x-4)(x-2)$

8.15 $(5t+1)(2t-3)$

8.17 $(x-8)(x+8)$

8.19 $(x-3)(x+3)(x^2+9)$

8.21 $(5y-3)^2$

8.23 $(2m-5)(4m^2+10m+25)$

8.25 $2x(x+7)(x-3)$

8.27 $5y(x-1)(x+1)(x^2+1)$

8.29 $(x+3-y)(x+3+y)$

8.31 $x = -2, 7$

8.33 $x = \frac{1}{2}, -3$

8.35 $x = -5, 3$

8.37 $x = -4, 3$

8.39 8 cm by 12 cm

8.41 5 cm and 12 cm

8.43 The product equals 10, not 0, so neither factor need be 10. Solving $x^2 - 3x - 10 = (x-5)(x+2) = 0$ gives $x = 5$ or $x = -2$.

8.45 Such a factorization would make $x^2 + 1 = 0$ at $x = r$, but $x^2 + 1 \geq 1$ for every real x .

8.47 $x^2 + 2x - 15 = 0$

8.49 $n^3 - n = (n-1)n(n+1)$, a product of three consecutive integers, one of which is divisible by 3 and at least one by 2.

CHAPTER 9

QUADRATIC EQUATIONS AND FUNCTIONS

9.1 $x = \pm 7$

9.3 $x = 3, -7$

9.5 $25; (x+5)^2$

9.7 $x = 3 \pm \sqrt{5}$

9.9 $x = -3 \pm \frac{\sqrt{46}}{2}$

9.11 $x = \frac{7 \pm \sqrt{33}}{4}$

9.13 $x = 1 \pm \sqrt{2}$

9.15 No real solutions

9.17 (a) One; (b) none; (c) two

9.19 $x = 2 \pm \sqrt{7}$

9.21 $(x-3)^2 - 4$; vertex $(3, -4)$

9.23 $3(x+1)^2 - 2$; vertex $(-1, -2)$

9.25 Vertex $(1, 4)$; x -intercepts $-1, 3$; y -intercept 3

9.27 $f(x) = (x-2)^2 - 1$

9.29 40 ft at $t = 1.5$ s

9.31 225 m^2

9.33 $(-1, 1)$ and $(3, 9)$

9.35 Only $(3, 1)$: the line is tangent to the parabola.

9.37 $(-\infty, -5] \cup [3, \infty)$

9.39 All real numbers

9.41 About 6.55 cm and 7.55 cm

9.43 Adding the two roots, the $\pm$ terms cancel: $\frac{-2b}{2a} = -\frac{b}{a}$. Multiplying, $\frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$.

9.47 A square of a real number is never negative.

9.49 $\frac{1 + \sqrt{5}}{2} \approx 1.618$

CHAPTER 10

RATIONAL EXPRESSIONS AND EQUATIONS

10.1 $x \neq 2, 3$

10.3 $\frac{x-4}{x+4}$

10.5 $\frac{x-2}{x+3}$

10.7 $\frac{x^2 + 2x + 4}{x + 2}$

10.9 $\frac{x + 1}{x - 1}$

10.11 $\frac{a}{b}$

10.13 $\frac{2x - 11}{(x + 2)(x - 1)}$

10.15 $\frac{2x + 1}{(x - 1)(x + 1)}$

10.17 $\frac{x}{x + 1}$

10.19 $\frac{ab}{3a + 2b}$

10.21 $x = 12$

10.23 $x = 7$

10.25 No solution ($x = 2$ is extraneous)

10.27 $x = 2, 3$

10.29 $R = \frac{R_1 R_2}{R_1 + R_2}$

10.31 6 hours

10.33 7.5 hours

10.35 3 mph

10.37 25

10.39 400 ft

10.41 Only common factors cancel; 5 is a term of $x + 5$, not a factor. At $x = 5$: $2 \neq 6$.

10.43 Multiplying by an expression that can be zero may add solutions. Example: Example 10.9.

10.45 It is halved; it becomes one third as large.

10.47 $A = 1$, $B = -1$; the sum telescopes to $1 - \frac{1}{100} = \frac{99}{100}$.

CHAPTER 11

RADICALS AND RATIONAL EXPONENTS

11.1 11; -4; 3; -2

11.3 $|x|$; x

11.5 $10\sqrt{2}$

11.7 $4ab^2\sqrt{3a}$

11.9 $\frac{\sqrt{7}}{4}$

11.11 $2\sqrt{2}$

11.13 $3\sqrt{10}$

11.15 $10 + \sqrt{2}$

11.17 $21 - 4\sqrt{5}$

11.19 $\frac{\sqrt{3}}{3}$

11.21 $\frac{3 + \sqrt{5}}{2}$

11.23 $\sqrt[4]{x^3}$; $y^{2/3}$

11.25 a^6

11.27 $\sqrt[4]{x^3}$

11.29 $x = 8$ ($x = 1$ is extraneous)

11.31 $x = 14$

11.33 $x = \pm 64$

11.35 $-i$; 1 ; $-i$

11.37 $3 + 3i$; $-3 + 5i$

11.39 $-21 - 20i$

11.41 $2 + i$

11.43 $x = 3 \pm i$

11.45 $\sqrt{\quad}$ is never negative; $x = -3$ gives $\sqrt{9} = 3 \neq -3$.

11.47 $\sqrt{-4}\sqrt{-9} = (2i)(3i) = -6$, but $\sqrt{36} = 6$. The product rule requires the roots to be real.

11.49 $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$

11.51 $2 + \sqrt{3}$

11.53 $z = \pm \frac{\sqrt{2}}{2}(1 + i)$

CHAPTER 12

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

12.1 $\frac{1}{9}$; 1; 9; $\sqrt{3}$

12.3 Growth; decay; decay; growth

12.5 \$10,798.87

12.7 5400

12.9 $\log_2 64 = 6$; $\log 0.01 = -2$; $\ln 1 = 0$

12.11 4 ; -3 ; 3 ; $\frac{3}{2}$

12.13 $(5, \infty)$; $(-2, 2)$

12.15 $2 \ln x + \frac{1}{2} \ln y - \ln z$

12.17 $\log_3(xy^2)$

12.19 $\log 10 = 1$

12.21 $x = 6$

12.23 $x = \frac{\ln 40}{\ln 5} \approx 2.292$

12.25 $x = 4$

12.27 $x = 81$

12.29 $x = e^3 + 1 \approx 21.09$

12.31 $x = \frac{2}{9}$

12.33 About 36.62 years

12.35 $\text{pH} \approx 3.49$; $[\text{H}^+] \approx 3.98 \times 10^{-8}$

12.37 A negative base gives non-real values such as $(-4)^{1/2}$; base 1 gives the constant function 1.

12.41 $\log(1+1) = \log 2 \approx 0.301$, but $\log 1 + \log 1 = 0$.

12.43 $b^y = x$ gives $y \ln b = \ln x$, so $y = \frac{\ln x}{\ln b}$.

12.45 $x = 9$