



CALCULUS I & II

LIMITS, DERIVATIVES, INTEGRALS, AND SERIES

FORGED MATH SERIES · VOLUME III

FORGED OFFICIAL TEXTBOOK

A thirteen-chapter text on single-variable calculus: limits, derivatives, integrals, differential equations, parametric and polar curves, and infinite series, developed with proofs.

First edition · Theorems, proofs, worked examples, and verified exercises

OMNISTRATA, LLC

PART I Differential Calculus

PART II Integral Calculus

PART III Curves and Series



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ForgEd Official Textbook · First Edition

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Calculus I & II

ForgEd Math Series, Volume III

First Edition

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The Forged Math Series

Volume I Algebra

Volume II Precalculus & Trigonometry

Volume III Calculus I–II

Volume IV Multivariable Calculus & Quantum Mechanics

Numerical results in the text, worked problems, and answers were checked by computer algebra and numerical computation.

PREFACE

This book is Volume III of the Forged Math Series. It develops the calculus of functions of one variable: limits and continuity, the derivative and its applications, the integral and its applications, differential equations, parametric and polar curves, and infinite series. It is written for a reader who has finished algebra and trigonometry (Volumes I and II) and is ready for a course that treats calculus as mathematics: definitions are stated precisely, the central theorems are proved, and the exercises are meant to take time.

HOW THE BOOK IS ORGANIZED

Part I (Chapters 1–6) is differential calculus: functions, limits and continuity, the derivative and the rules for computing it, the Mean Value Theorem and the shape of graphs, and the uses of the derivative to optimize and approximate. **Part II** (Chapters 7–10) is integral calculus: the definite integral and the Fundamental Theorem of Calculus, applications of integration to geometry and physics, techniques of integration, and first-order differential equations. **Part III** (Chapters 11–13) covers parametric and polar curves, infinite sequences and series, and power series. Chapters 1–8 correspond to a first semester (Calculus I), and Chapters 9–13 to a second (Calculus II). Three appendices give proofs of the foundational theorems, reference tables, and numerical methods with working Python programs.

PREREQUISITES

Fluency with algebra, functions, exponentials and logarithms, and trigonometry, at the level of Volumes I and II. Chapter 1 reviews the essentials, with emphasis on the points that calculus depends on.

HOW TO READ THIS BOOK

Each chapter opens with its learning objectives. Boxed material comes in four kinds.

Definition 0.1 (Definitions, theorems, propositions). Shaded boxes contain the statements you should be able to reproduce from memory. Theorems carry a gold rule.

Example 0.2 (Examples). Worked examples. Cover the solution and attempt the problem before reading it.

COMMON ERROR. Black-ruled boxes describe the specific mistakes students make most often, and how to detect them.

LOOKING AHEAD. Dark boxes connect the current material to where it is used later, often in Volume IV.

Theorems, definitions, and examples share one numbering sequence within each chapter, so Example 3.7 follows Theorem 3.6.

WORKED PROBLEMS, EXERCISES, AND ANSWERS

Every chapter closes with a section of *Worked Problems*, just before the exercises: longer problems, solved in full, that combine several ideas from the chapter and show the reasoning a careful solver would record. Exercises are grouped as *Computation*, *Theory and Proof*, and *Challenge*. Answers to odd-numbered computational exercises appear at the back of the book. Every numerical answer in the examples, the worked problems, and the answer section was checked by computer algebra or numerical computation before publication; if your result disagrees, find the step where it diverges.

NOTATION

Angles are in radians unless degrees are marked. $\ln x$ is the natural logarithm and $\log_b x$ the logarithm to base b . The inverse trigonometric functions are written $\arcsin$, $\arccos$, $\arctan$, never $\sin^{-1}$, to avoid confusion with reciprocals. Leib-

niz notation $\frac{dy}{dx}$ and prime notation $f'(x)$ are used interchangeably.

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PART I

DIFFERENTIAL CALCULUS

Functions, limits, continuity, the derivative, and what the derivative reveals about the shape and behavior of functions.

CHAPTER 1

FUNCTIONS AND THEIR GRAPHS

LEARNING OBJECTIVES

- Work with the real numbers, intervals, absolute value, and inequalities, and state the completeness axiom.
- Use the definition of a function, find domains and ranges, and recognize even, odd, and monotone functions.
- Build new functions by transformation, composition, and inversion, and find inverse functions.
- Know the algebraic, exponential, logarithmic, trigonometric, inverse trigonometric, and hyperbolic functions, their graphs, and their key identities.

Calculus is the mathematics of change, and what changes is described by functions. This chapter assembles the functions that the rest of the book differentiates and integrates, and fixes the language for discussing them. Much of it will be familiar from algebra and trigonometry (Volumes I and II). Read it anyway: several points that earlier courses treat informally, the completeness of the real numbers, the precise meaning of an inverse function, and the restricted domains of the inverse trigonometric functions, are exactly the points on which calculus depends.

1.1 THE REAL NUMBERS

The number systems of calculus are nested:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \subset \mathbb{Q} \subset \mathbb{R}.$$

The rational numbers $\mathbb{Q}$ are the quotients p/q of integers with $q \neq 0$. Their decimal expansions terminate or repeat. The real numbers $\mathbb{R}$ fill in the gaps: every point of a line, once an origin and a unit are chosen, corresponds to exactly

one real number. Real numbers that are not rational, such as $\sqrt{2}$, π , and e , are *irrational*.

Theorem 1.1 ($\sqrt{2}$ is irrational). *There is no rational number r with $r^2 = 2$.*

Proof. Suppose $r = p/q$ with p, q integers having no common factor, and $r^2 = 2$. Then $p^2 = 2q^2$, so p^2 is even, so p is even (the square of an odd number is odd). Write $p = 2k$. Then $4k^2 = 2q^2$, so $q^2 = 2k^2$ is even and q is even. Now p and q share the factor 2, a contradiction. ■

1.1.1 ORDER, INTERVALS, AND ABSOLUTE VALUE

The real numbers are ordered: for any a, b exactly one of $a < b$, $a = b$, $a > b$ holds. Inequalities may be added, and multiplied by positive numbers; multiplying by a negative number reverses them. Sets of real numbers between two endpoints are *intervals*: $(a, b) = \{x : a < x < b\}$ is open, $[a, b] = \{x : a \leq x \leq b\}$ is closed, and $[a, b)$, $(a, b]$ are half-open. Unbounded intervals use ∞ : $[a, \infty) = \{x : x \geq a\}$, and $(-\infty, \infty) = \mathbb{R}$. The symbol ∞ is not a number; it records that the interval has no endpoint on that side.

Definition 1.2 (Absolute value). $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Equivalently $|x| = \sqrt{x^2}$, and $|a - b|$ is the distance between a and b on the line.

Theorem 1.3 (Properties of absolute value). *For all real a, b and $\delta > 0$:*

- (a) $|ab| = |a||b|$;
- (b) $|x| < \delta$ if and only if $-\delta < x < \delta$;
- (c) $|a + b| \leq |a| + |b|$ (the triangle inequality);
- (d) $||a| - |b|| \leq |a - b|$ (the reverse triangle inequality).

Proof. (a) Square both sides: $(ab)^2 = a^2b^2$, and take nonnegative square roots. (b) If $x \geq 0$ the condition is $x < \delta$; if $x < 0$ it is $-x < \delta$. Together these say $-\delta < x < \delta$. (c) Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, adding gives $-(|a| + |b|) \leq a + b \leq |a| + |b|$, which is $|a + b| \leq |a| + |b|$ by (b) (with $\leq$ in place of $<$). (d) By (c), $|a| = |(a - b) + b| \leq |a - b| + |b|$, so $|a| - |b| \leq |a - b|$. Exchanging a and b gives $|b| - |a| \leq |a - b|$. ■

The triangle inequality is the single most used inequality in analysis. Part (b) translates statements about distance into statements about intervals: $|x - c| < \delta$ says x lies in $(c - \delta, c + \delta)$.

Example 1.4 (Solving inequalities). Solve (a) $|2x - 3| \leq 5$ and (b) $\frac{x+1}{x-2} \geq 3$.

Solution. (a) By Theorem 1.3(b), $-5 \leq 2x - 3 \leq 5$, so $-2 \leq 2x \leq 8$ and $x \in [-1, 4]$.

(b) Do not multiply by $x - 2$, whose sign is unknown. Instead move everything to one side: $\frac{x+1}{x-2} - 3 = \frac{x+1-3(x-2)}{x-2} = \frac{7-2x}{x-2} \geq 0$. The numerator vanishes at $x = \frac{7}{2}$ and the denominator at $x = 2$. A sign chart on the intervals $(-\infty, 2)$, $(2, \frac{7}{2})$, $(\frac{7}{2}, \infty)$ shows the quotient is negative, positive, negative. Including the zero at $\frac{7}{2}$ and excluding 2, the solution is $(2, \frac{7}{2}]$.

COMMON ERROR. Multiplying both sides of an inequality by an expression whose sign is not known. In (b) above, multiplying by $x - 2$ gives $x + 1 \geq 3x - 6$, so $x \leq \frac{7}{2}$, which wrongly includes every $x < 2$. If you must multiply, split into the cases $x - 2 > 0$ and $x - 2 < 0$.

1.1.2 COMPLETENESS

What distinguishes $\mathbb{R}$ from $\mathbb{Q}$ is not visible in any finite computation, and yet every major theorem of calculus depends on it. A set $S \subseteq \mathbb{R}$ is *bounded above* if there is a number M with $s \leq M$ for all $s \in S$; such an M is an *upper bound*. The *least upper bound* or *supremum*, $\sup S$, is an upper bound that is smaller than every other upper bound.

Definition 1.5 (The completeness axiom). Every nonempty set of real numbers that is bounded above has a least upper bound in $\mathbb{R}$.

The rationals fail this. The set $S = \{r \in \mathbb{Q} : r^2 < 2\}$ is bounded above (by 2), but it has no least upper bound *in* $\mathbb{Q}$: any rational upper bound M has $M^2 > 2$ (since $M^2 = 2$ is impossible by Theorem 1.1), and then a slightly smaller rational is still an upper bound. In $\mathbb{R}$, $\sup S = \sqrt{2}$. Completeness guarantees that the real line has no holes, and it is what makes the Intermediate Value Theorem, the Extreme Value Theorem, and the convergence of monotone sequences true. Proofs of those theorems from the axiom are given in Appendix A.

Example 1.6 (A supremum). Find $\sup S$ for $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Is it attained?

Solution. $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$, and every element is less than 1, so 1 is an upper bound. No smaller number $M < 1$ is an upper bound: choose $n > \frac{1}{1-M}$; then $1 - \frac{1}{n} > M$. So $\sup S = 1$. It is not attained, since $1 - \frac{1}{n} < 1$ for every n . A supremum need not belong to the set; when it does, it is the *maximum*.

1.2 FUNCTIONS

Definition 1.7 (Function). A *function* f from a set D to a set E is a rule that assigns to each element $x \in D$ exactly one element $f(x) \in E$. The set D is the *domain* of f , and the set $\{f(x) : x \in D\}$ of values actually taken is its *range*.

In this book D and E are sets of real numbers unless stated otherwise. When a function is given by a formula and no domain is stated, the domain is the largest set of real numbers for which the formula makes sense. The *graph* of f is the set $\{(x, f(x)) : x \in D\}$ in the plane. A curve in the plane is the graph of a function of x exactly when no vertical line meets it more than once (the *vertical line test*), since each x must have exactly one value.

Example 1.8 (Domains and ranges). Find the domain and range of (a) $f(x) = \sqrt{4 - x^2}$, (b) $g(x) = \frac{1}{x^2 - x - 6}$, (c) $h(x) = \ln(\ln x)$.

Solution. (a) We need $4 - x^2 \geq 0$, so the domain is $[-2, 2]$. The values $4 - x^2$ fill $[0, 4]$, so the range is $[0, 2]$. The graph is the upper half of the circle $x^2 + y^2 = 4$.

(b) $x^2 - x - 6 = (x - 3)(x + 2)$, so the domain is $\mathbb{R} \setminus \{-2, 3\}$. On $(-2, 3)$ the denominator is negative with minimum $-\frac{25}{4}$ at $x = \frac{1}{2}$, so g takes all values in $(-\infty, -\frac{4}{25}]$ there. Outside, the denominator takes all positive values, so g takes all of $(0, \infty)$. The range is $(-\infty, -\frac{4}{25}] \cup (0, \infty)$.

(c) $\ln x$ requires $x > 0$, and $\ln(\ln x)$ requires $\ln x > 0$, that is, $x > 1$. The domain is $(1, \infty)$. As x ranges over $(1, \infty)$, $\ln x$ takes every value in $(0, \infty)$, and so $\ln(\ln x)$ takes every real value. The range is $\mathbb{R}$.

1.2.1 SYMMETRY AND MONOTONICITY

Definition 1.9 (Even, odd, increasing, decreasing). A function f is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$ for all x in its (symmetric) domain. It is *increasing* on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I , and *decreasing* if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

The graph of an even function is symmetric about the y -axis, and that of an odd function about the origin. The names come from powers: x^n is even when n is even and odd when n is odd. Every function on a symmetric domain is uniquely the sum of an even and an odd function:

$$f(x) = \underbrace{\frac{1}{2}(f(x) + f(-x))}_{\text{even}} + \underbrace{\frac{1}{2}(f(x) - f(-x))}_{\text{odd}}.$$

For $f(x) = e^x$ the two parts are the hyperbolic cosine and sine of Section 1.7.

1.2.2 PIECEWISE FUNCTIONS

Functions need not be given by a single formula. The *absolute value* function is piecewise, and so are the *floor* function $\lfloor x \rfloor$, the greatest integer $\leq x$, and the *Heaviside step* $H(x)$, which is 0 for $x < 0$ and 1 for $x \geq 0$. These functions, which jump or have corners, will be our main examples of the ways a function can fail to be continuous or differentiable.

1.3 NEW FUNCTIONS FROM OLD

1.3.1 TRANSFORMATIONS

Given the graph of $y = f(x)$ and a constant $c > 0$:

Formula	Effect on the graph	Formula	Effect on the graph
$f(x) + c$	shift up by c	$f(x - c)$	shift right by c
$f(x) - c$	shift down by c	$f(x + c)$	shift left by c
$cf(x)$	stretch vertically by c	$f(cx)$	compress horizontally by c
$-f(x)$	reflect in the x -axis	$f(-x)$	reflect in the y -axis

The horizontal effects run opposite to intuition: $f(x - c)$ moves right because the graph reaches the value $f(0)$ at $x = c$ instead of at $x = 0$. The same reasoning explains why $f(2x)$ compresses: the graph reaches $f(1)$ already at $x = \frac{1}{2}$.

Example 1.10 (Completing the square). Describe the graph of $y = 2x^2 - 12x + 13$ as a transformation of $y = x^2$.

Solution. $2x^2 - 12x + 13 = 2(x^2 - 6x + 9) - 18 + 13 = 2(x - 3)^2 - 5$. Start with $y = x^2$, shift right 3, stretch vertically by 2, and shift down 5. The vertex is $(3, -5)$ and the parabola opens upward.

1.3.2 COMPOSITION

Definition 1.11 (Composition). The *composition* of f and g is $(f \circ g)(x) = f(g(x))$. Its domain is the set of x in the domain of g for which $g(x)$ is in the domain of f .

Composition is not commutative. With $f(x) = \sqrt{x}$ and $g(x) = x - 4$: $(f \circ g)(x) = \sqrt{x - 4}$ has domain $[4, \infty)$, while $(g \circ f)(x) = \sqrt{x} - 4$ has domain $[0, \infty)$. Recognizing a function as a composition, $\sin(x^2) = f(g(x))$ with $f = \sin$ and $g(x) = x^2$, is the skill that the chain rule (Chapter 4) and substitution (Chapter 7) depend on.

1.3.3 INVERSE FUNCTIONS

Definition 1.12 (One-to-one; inverse). A function f is *one-to-one* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. If f is one-to-one with domain A and range B , its *inverse* f^{-1} has domain B and range A , and is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

Then $f^{-1}(f(x)) = x$ for $x \in A$ and $f(f^{-1}(y)) = y$ for $y \in B$. The notation f^{-1} does not mean $1/f$. A function is one-to-one exactly when no horizontal line meets its graph more than once. Since (a, b) is on the graph of f exactly when (b, a) is on the graph of f^{-1} , the graph of f^{-1} is the reflection of the graph of f in the line $y = x$. A function that is increasing (or decreasing) on its whole domain is one-to-one.

Example 1.13 (Finding an inverse). Find the inverse of $f(x) = \frac{2x + 1}{x - 3}$, and its domain and range.

Solution. Solve $y = \frac{2x+1}{x-3}$ for x : $y(x - 3) = 2x + 1$, so $x(y - 2) = 3y + 1$

and $x = \frac{3y+1}{y-2}$. Every $y \neq 2$ comes from exactly one x , so f is one-to-one with range $\mathbb{R} \setminus \{2\}$. Writing the variable as x again, $f^{-1}(x) = \frac{3x+1}{x-2}$, with domain $\mathbb{R} \setminus \{2\}$ and range $\mathbb{R} \setminus \{3\}$, the domain of f . Check: $f^{-1}(f(0)) = f^{-1}(-\frac{1}{3}) = \frac{-\frac{1}{3}+1}{-\frac{1}{3}-2} = 0$.

When f is not one-to-one, we restrict its domain to an interval where it is, and invert the restriction. That is how $\sqrt{x}$ arises as the inverse of x^2 on $[0, \infty)$, and how the inverse trigonometric functions are defined below.

1.4 POLYNOMIAL, RATIONAL, AND ALGEBRAIC FUNCTIONS

A *polynomial* is $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$; if $a_n \neq 0$ its *degree* is n . A *rational function* is a quotient p/q of polynomials, defined where $q \neq 0$. An *algebraic function* is one built from polynomials by the four arithmetic operations and taking roots, such as $\sqrt{1+x^3}/(x-1)^{2/3}$. Functions that are not algebraic, such as e^x , $\ln x$, and $\sin x$, are *transcendental*.

Theorem 1.14 (Factor theorem). *If p is a polynomial and $p(c) = 0$, then $p(x) = (x - c)q(x)$ for a polynomial q of degree one less. Consequently a nonzero polynomial of degree n has at most n real roots.*

Proof. For each $k \geq 1$, $x^k - c^k = (x - c)(x^{k-1} + cx^{k-2} + \cdots + c^{k-1})$. So $p(x) - p(c) = \sum a_k(x^k - c^k)$ is divisible by $x - c$, and $p(c) = 0$. Each root removes one factor of $x - c$ and lowers the degree by one, so there can be at most n of them. ■

The behavior of a polynomial for large $|x|$ is governed by its leading term: $p(x) = a_n x^n (1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n})$, and the parenthesis tends to 1. This simple factoring idea will settle limits at infinity of rational functions in Chapter 2.

Example 1.15 (Power functions). The graphs of x^n for positive integers n all pass through $(0, 0)$ and $(1, 1)$. On $(0, 1)$ higher powers are smaller, $x^3 < x^2 < x$, while on $(1, \infty)$ they are larger. The reciprocals x^{-n} have vertical asymptotes at 0. Fractional powers $x^{p/q}$ are defined for $x \geq 0$, and for $x < 0$ as well when q is odd; thus $x^{1/3}$ is defined on all of $\mathbb{R}$, but $x^{1/2}$ only on $[0, \infty)$.

1.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

For $b > 0$, the power b^x is defined first for integers, then for rationals by $b^{p/q} = \sqrt[q]{b^p}$, and finally for irrationals by continuity: $b^{\sqrt{2}}$ is the limit of $b^{1.4}, b^{1.41}, b^{1.414}, \dots$ (the completeness axiom guarantees the limit exists). The result is the *exponential function with base b* . For all real x, y :

$$b^{x+y} = b^x b^y, \quad b^{x-y} = \frac{b^x}{b^y}, \quad (b^x)^y = b^{xy}, \quad (ab)^x = a^x b^x.$$

For $b > 1$, b^x increases from 0 (as $x \rightarrow -\infty$) to ∞ ; for $0 < b < 1$ it decreases. Either way its range is $(0, \infty)$.

Among all bases one is special. In Chapter 3 we will show that there is exactly one base e for which the graph of b^x has slope 1 at $x = 0$, and that

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\,281\,828\,\dots$$

The *natural exponential function* is e^x , written also $\exp(x)$.

Definition 1.16 (Logarithm). For $b > 0$, $b \neq 1$, $\log_b x$ is the inverse of b^x : $\log_b x = y \iff b^y = x$. Its domain is $(0, \infty)$ and its range is $\mathbb{R}$. The *natural logarithm* is $\ln x = \log_e x$.

The laws of exponents become the laws of logarithms:

$$\ln(xy) = \ln x + \ln y, \quad \ln \frac{x}{y} = \ln x - \ln y, \quad \ln(x^r) = r \ln x, \quad \log_b x = \frac{\ln x}{\ln b}.$$

The last, the *change of base* formula, follows by taking $\ln$ of $b^y = x$. Every exponential can be written with base e : $b^x = e^{x \ln b}$. So e^x and $\ln x$ are the only exponential and logarithm we will ever really need.

Example 1.17 (Exponential models). Carbon-14 decays with half-life 5730 years. What fraction remains after 10 000 years? How old is a sample with 20% of its original carbon-14?

Solution. The amount is $A(t) = A_0 \left(\frac{1}{2}\right)^{t/5730} = A_0 e^{-kt}$ with $k = \frac{\ln 2}{5730} = 1.2097 \times 10^{-4}$ per year. After 10 000 years, $A/A_0 = e^{-1.2097} = 0.298$, about 30%. For $A/A_0 = 0.2$: $t = \frac{\ln 5}{k} = \frac{5730 \ln 5}{\ln 2} = 13\,300$ years.

COMMON ERROR. $\ln(a + b)$ is not $\ln a + \ln b$, and $(\ln x)^2$ is not $\ln(x^2) = 2 \ln x$. Logarithms turn *products* into sums. Likewise $e^{a+b} = e^a e^b$, but $e^{ab} \neq e^a e^b$.

1.6 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

In calculus, angles are measured in *radians*: the radian measure of an angle is the length of the arc it cuts from a circle of radius 1. A full turn is 2π radians, so $180^\circ = \pi$ rad. Only in radians is $\frac{d}{dx} \sin x = \cos x$ (Chapter 3); in degrees an annoying factor $\frac{\pi}{180}$ would appear in every formula.

For any real θ , the point at arc length θ counterclockwise from $(1, 0)$ on the unit circle is $(\cos \theta, \sin \theta)$. Then $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\sec \theta = \frac{1}{\cos \theta}$, $\csc \theta = \frac{1}{\sin \theta}$, $\cot \theta = \frac{\cos \theta}{\sin \theta}$, wherever the denominators are nonzero. The sine and cosine have period 2π ; the tangent and cotangent have period π .

Theorem 1.18 (Trigonometric identities). *For all α, β, θ where defined:*

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1, & 1 + \tan^2 \theta &= \sec^2 \theta, & 1 + \cot^2 \theta &= \csc^2 \theta, \\ \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, & \sin 2\theta &= 2 \sin \theta \cos \theta, \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta, \\ \cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta), & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta). \end{aligned}$$

These were derived in Volume II. The last two, the *half-angle* or *power-reduction* formulas, will be indispensable for integrating $\sin^2 x$ and $\cos^2 x$ in Chapter 9.

1.6.1 INVERSE TRIGONOMETRIC FUNCTIONS

The trigonometric functions are periodic, so they are not one-to-one. To invert them we restrict each to an interval on which it is one-to-one and takes every value of its range once:

Definition 1.19 (Inverse trigonometric functions).

$$\begin{aligned}\arcsin x = y &\iff \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, & \text{domain } [-1, 1], \\ \arccos x = y &\iff \cos y = x \text{ and } 0 \leq y \leq \pi, & \text{domain } [-1, 1], \\ \arctan x = y &\iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}, & \text{domain } \mathbb{R}.\end{aligned}$$

The notations $\sin^{-1} x$ and so on are common, but they invite the confusion $\sin^{-1} x = 1/\sin x$, which is false. Note that $\arcsin(\sin x) = x$ only for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$: for example, $\arcsin(\sin \pi) = \arcsin 0 = 0$.

Example 1.20 (Composing trigonometric and inverse functions). Simplify $\sin(\arccos x)$ and $\tan(\arcsin x)$.

Solution. Let $\theta = \arccos x$, so $\cos \theta = x$ and $0 \leq \theta \leq \pi$. On this interval $\sin \theta \geq 0$, so $\sin \theta = +\sqrt{1 - \cos^2 \theta} = \sqrt{1 - x^2}$. Thus $\sin(\arccos x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$.

Let $\theta = \arcsin x$, so $\sin \theta = x$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, where $\cos \theta \geq 0$. Then $\cos \theta = \sqrt{1 - x^2}$ and $\tan(\arcsin x) = \frac{x}{\sqrt{1 - x^2}}$ for $-1 < x < 1$. A right triangle with hypotenuse 1 and opposite side x gives the same results, and is the standard way to remember them. These simplifications are the heart of trigonometric substitution in Chapter 9.

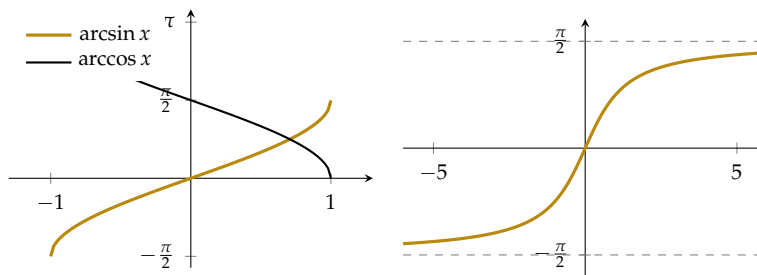


Figure 1.1: The inverse trigonometric functions. The arctangent has horizontal asymptotes $y = \pm\frac{\pi}{2}$.

1.7 HYPERBOLIC FUNCTIONS

Definition 1.21 (Hyperbolic functions).

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

These are the even and odd parts of e^x . They satisfy identities parallel to the trigonometric ones, usually with a change of sign:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1, \\ \sinh 2x &= 2 \sinh x \cosh x, \quad \cosh 2x = \cosh^2 x + \sinh^2 x. \end{aligned}$$

The first, verified by expanding, says that $(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$, just as $(\cos t, \sin t)$ lies on the circle $x^2 + y^2 = 1$: hence the name. A hanging chain or cable takes the shape of a *catenary* $y = a \cosh(x/a)$, and $\tanh$ describes the velocity of a falling body with air resistance proportional to the square of its speed.

Example 1.22 (Inverse hyperbolic sine). Show that $\sinh$ is one-to-one on $\mathbb{R}$ and that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

Solution. e^x increases and e^{-x} decreases, so $\sinh x = \frac{1}{2}(e^x - e^{-x})$ increases and is one-to-one. Solve $x = \frac{e^y - e^{-y}}{2}$ for y : with $u = e^y > 0$, $2x = u - \frac{1}{u}$, so $u^2 - 2xu - 1 = 0$ and $u = x \pm \sqrt{x^2 + 1}$. Since $\sqrt{x^2 + 1} > |x|$, only the plus sign gives $u > 0$. Hence $y = \ln(x + \sqrt{x^2 + 1})$.

LOOKING AHEAD. The hyperbolic and trigonometric functions are the same functions in disguise. In Chapter 13 we will see that $e^{i\theta} = \cos \theta + i \sin \theta$ for real θ , which gives $\cos \theta = \cosh(i\theta)$ and $\sin \theta = -i \sinh(i\theta)$. Every identity on one side becomes an identity on the other. This connection is the starting point of complex analysis in Volume IV.

1.8 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— **Problem 1.1** An inequality with two absolute values

Solve $|x - 1| + |x + 2| \leq 5$, and interpret the answer geometrically.

Solution. The expressions inside the absolute values change sign at $x = 1$ and $x = -2$, which split the line into three intervals.

- For $x < -2$: both are negative, and the inequality is $-(x - 1) - (x + 2) \leq 5$, that is, $-2x - 1 \leq 5$, so $x \geq -3$. This gives $[-3, -2)$.
- For $-2 \leq x < 1$: it is $-(x - 1) + (x + 2) = 3 \leq 5$, always true. This gives $[-2, 1)$.
- For $x \geq 1$: it is $(x - 1) + (x + 2) = 2x + 1 \leq 5$, so $x \leq 2$. This gives $[1, 2]$.

The union is $[-3, 2]$. Geometrically, $|x - 1| + |x + 2|$ is the sum of the distances from x to 1 and to -2 . For any point between them the sum is the distance between them, 3. Outside, the sum grows by twice the distance beyond the nearer endpoint, and reaches 5 when that distance is 1: at $x = -3$ and $x = 2$. In the plane, the same condition on distances to two points defines an ellipse, and this interval is its one-dimensional version.

— **Problem 1.2** Irrationality, twice

Prove that $\sqrt{3}$ is irrational, and then that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution. Suppose $\sqrt{3} = p/q$ in lowest terms. Then $p^2 = 3q^2$, so 3 divides p^2 . Since 3 is prime, 3 divides p (if a prime divides a product, it divides a factor). Write $p = 3k$: then $9k^2 = 3q^2$, so $q^2 = 3k^2$ and 3 divides q , contradicting lowest terms.

For the sum, suppose $s = \sqrt{2} + \sqrt{3}$ were rational. Squaring, $s^2 = 5 + 2\sqrt{6}$, so $\sqrt{6} = \frac{s^2 - 5}{2}$ would be rational. But $\sqrt{6}$ is irrational by the same argument with the prime 2: $p^2 = 6q^2$ forces p even, then q even. Hence s is irrational. The strategy, reducing a claim about a complicated number to one about a simpler number by algebra, is typical: rational numbers are closed under the four arithmetic operations, so any rational expression of a rational number is rational.

— **Problem 1.3** The domain of a composition

Let $f(x) = \sqrt{x - 1}$ and $g(x) = \frac{1}{x - 3}$. Find $f \circ g$ and $g \circ f$ with their domains.

Solution. $(f \circ g)(x) = \sqrt{\frac{1}{x - 3} - 1} = \sqrt{\frac{4 - x}{x - 3}}$. The domain requires $x \neq 3$ (so g is defined) and $\frac{1}{x - 3} \geq 1$ (so f can be applied). The second condition forces

$x - 3 > 0$ and then $1 \geq x - 3$, so the domain is $(3, 4]$. Note that the simplified formula $\sqrt{\frac{4-x}{x-3}}$ gives the same domain here, but in general simplification can hide or add restrictions, so domains must be read from the original composition.

$(g \circ f)(x) = \frac{1}{\sqrt{x-1}-3}$. It requires $x \geq 1$ and $\sqrt{x-1} \neq 3$, that is, $x \neq 10$. The domain is $[1, 10) \cup (10, \infty)$.

— Problem 1.4 Inverting a restricted quadratic

Let $f(x) = x^2 - 4x + 3$ on the domain $[2, \infty)$. Show that f is one-to-one, and find f^{-1} with its domain and range.

Solution. Completing the square, $f(x) = (x - 2)^2 - 1$. On $[2, \infty)$, $x - 2 \geq 0$ increases, so $(x - 2)^2$ increases and f is one-to-one, with range $[-1, \infty)$. Solve $y = (x - 2)^2 - 1$: $(x - 2)^2 = y + 1$, and since $x - 2 \geq 0$ on the domain, $x - 2 = +\sqrt{y + 1}$. So

$$f^{-1}(x) = 2 + \sqrt{x + 1}, \quad \text{domain } [-1, \infty), \text{ range } [2, \infty).$$

The choice of the positive root was forced by the restricted domain. On $(-\infty, 2]$ the inverse would be $2 - \sqrt{x + 1}$. Check: $f(f^{-1}(x)) = (\sqrt{x + 1})^2 - 1 = x$.

— Problem 1.5 Exponential and logarithmic equations

Solve (a) $3^{2x} - 4 \cdot 3^x + 3 = 0$ and (b) $\log_2 x + \log_2(x - 2) = 3$.

Solution. (a) With $u = 3^x > 0$, the equation is $u^2 - 4u + 3 = (u - 1)(u - 3) = 0$, so $3^x = 1$ or $3^x = 3$: $x = 0$ or $x = 1$.

(b) The equation requires $x > 0$ and $x - 2 > 0$, so $x > 2$. Combining the logarithms, $\log_2(x(x - 2)) = 3$, so $x^2 - 2x = 8$ and $(x - 4)(x + 2) = 0$. The root $x = -2$ is outside the domain, since $\log_2(-2)$ is undefined, and the law $\log a + \log b = \log ab$ is valid only for positive a, b . The solution is $x = 4$: check, $\log_2 4 + \log_2 2 = 2 + 1 = 3$.

— Problem 1.6 A trigonometric equation and an identity

(a) Solve $2 \sin^2 x + 3 \sin x - 2 = 0$ for $0 \leq x < 2\pi$. (b) Prove that $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$. What is the sum for $x < 0$?

Solution. (a) Factor as a quadratic in $s = \sin x$: $2s^2 + 3s - 2 = (2s - 1)(s + 2) = 0$. Since $|\sin x| \leq 1$, only $\sin x = \frac{1}{2}$ is possible, giving $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

(b) Let $\theta = \arctan x$. For $x > 0$, $0 < \theta < \frac{\pi}{2}$, and then $\frac{\pi}{2} - \theta$ is also in $(0, \frac{\pi}{2})$, with $\tan(\frac{\pi}{2} - \theta) = \cot \theta = \frac{1}{x}$. Since $\frac{\pi}{2} - \theta$ lies in the range of $\arctan$, it equals $\arctan \frac{1}{x}$.

Hence the sum is $\frac{\pi}{2}$. For $x < 0$, both functions are odd, so $\arctan x + \arctan \frac{1}{x} = -(\arctan |x| + \arctan \frac{1}{|x|}) = -\frac{\pi}{2}$. The step “ $\frac{\pi}{2} - \theta$ lies in the range” is essential: an identity $\tan A = \tan B$ gives $A = B$ only when both are in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

— **Problem 1.7** The inverse hyperbolic tangent

Show that $\tanh$ is one-to-one from $\mathbb{R}$ onto $(-1, 1)$, and that $\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$.

Solution. Dividing numerator and denominator by e^x , $\tanh x = \frac{e^{2x}-1}{e^{2x}+1} = 1 - \frac{2}{e^{2x}+1}$. As x increases, $e^{2x} + 1$ increases, $\frac{2}{e^{2x}+1}$ decreases, and $\tanh x$ increases, so $\tanh$ is one-to-one. As $x \rightarrow \infty$, $\tanh x \rightarrow 1$, and as $x \rightarrow -\infty$, $\tanh x \rightarrow -1$; every value in between is taken (a fact that will follow from the Intermediate Value Theorem). Solving $y = \frac{u-1}{u+1}$ with $u = e^{2x}$ gives $u = \frac{1+y}{1-y}$, so $x = \frac{1}{2} \ln \frac{1+y}{1-y}$ for $-1 < y < 1$. In the theory of relativity, $\tanh^{-1}(v/c)$ is the *rapidity* of a particle with speed v , a quantity that, unlike speed, simply adds when velocities are combined.

EXERCISES

A. COMPUTATION

- 1.1 Solve $|3x + 2| < 7$ and $|x - 4| \geq 2$. Write each answer in interval notation.
- 1.2 Solve $\frac{2x-1}{x+3} \leq 1$.
- 1.3 Solve $x^2 - 5x + 4 < 0$ and $x^3 > x$.
- 1.4 Find $\sup S$ and $\inf S$ for $S = \{(-1)^n/n : n \in \mathbb{N}\}$. Are they attained?
- 1.5 Find the domain of $f(x) = \frac{\sqrt{x+2}}{x^2-9}$.
- 1.6 Find the domain and range of $f(x) = \sqrt{6-x-x^2}$.
- 1.7 Find the domain of $f(x) = \ln(x^2 - 4x)$.
- 1.8 Determine whether each function is even, odd, or neither: $x^3 - x$, $x^2 + \cos x$, $e^x + x$, $\frac{x}{1+x^4}$.
- 1.9 Write $f(x) = \frac{1}{1-x}$ as the sum of an even and an odd function on $(-1, 1)$.

- 1.10** Describe how to obtain the graph of $y = -2\sqrt{x+3} + 1$ from that of $y = \sqrt{x}$, and give its domain and range.
- 1.11** Write $y = 3x^2 + 12x + 7$ in vertex form and give the vertex.
- 1.12** For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$, find $f \circ g$, $g \circ f$, and their domains.
- 1.13** Express $h(x) = \cos^3(2x+1)$ as a composition of three simpler functions.
- 1.14** Find the inverse of $f(x) = \frac{x-1}{2x+3}$, with its domain and range.
- 1.15** Find the inverse of $f(x) = e^{2x-1} + 3$ and its domain.
- 1.16** Find the inverse of $f(x) = \sqrt{4-x}$ for $x \leq 4$.
- 1.17** Find all real roots of $x^3 - 6x^2 + 11x - 6$ using the factor theorem.
- 1.18** Simplify $\log_3 81 - \log_2 \frac{1}{8} + \ln e^5$.
- 1.19** Solve $e^{2x} - 5e^x + 6 = 0$.
- 1.20** Solve $\ln(x+1) - \ln(x-1) = \ln 3$.
- 1.21** Solve $2^{x+1} = 5^x$ exactly and to four decimal places.
- 1.22** A population grows from 2000 to 5000 in 6 years, exponentially. Find its doubling time.
- 1.23** Convert 135° and -20° to radians, and $\frac{7\pi}{6}$ to degrees.
- 1.24** Find all solutions in $[0, 2\pi)$ of $\cos 2x = \cos x$.
- 1.25** Find all solutions in $[0, 2\pi)$ of $\tan^2 x = 3$.
- 1.26** Evaluate $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$, $\arccos\left(-\frac{1}{2}\right)$, $\arctan(-1)$, and $\arccos\left(\cos \frac{5\pi}{3}\right)$.
- 1.27** Simplify $\cos(\arctan x)$ and $\sec(\arcsin x)$.
- 1.28** Evaluate $\sin\left(2\arccos \frac{3}{5}\right)$ exactly.
- 1.29** Compute $\cosh(\ln 3)$ and $\sinh(\ln 2)$ exactly.
- 1.30** Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$, where $\cosh$ is restricted to $[0, \infty)$.

B. THEORY AND PROOF

- 1.31** Prove that the product of a nonzero rational number and an irrational number is irrational.

- 1.32** Prove that between any two distinct real numbers there is a rational number. (Use the fact that for any $h > 0$ there is $n \in \mathbb{N}$ with $nh > 1$.)
- 1.33** Prove that $|a - c| \leq |a - b| + |b - c|$ for all real a, b, c .
- 1.34** Prove that if S is nonempty and bounded above, then $M = \sup S$ satisfies: for every $\epsilon > 0$ there is $s \in S$ with $s > M - \epsilon$.
- 1.35** Prove that the composition of two odd functions is odd, and that the product of two odd functions is even.
- 1.36** Prove that the composition of two increasing functions is increasing.
- 1.37** Prove that if f is increasing on its domain, then f is one-to-one and f^{-1} is increasing.
- 1.38** Derive $\cosh^2 x - \sinh^2 x = 1$ and $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ from the definitions.
- 1.39** Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$.

C. CHALLENGE

- 1.40** Prove that $\log_2 3$ is irrational.
- 1.41** Prove the AM–GM inequality $\sqrt{ab} \leq \frac{a+b}{2}$ for $a, b \geq 0$, with equality iff $a = b$. Deduce that among rectangles of given perimeter the square has the largest area.
- 1.42** Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that are both even and odd.
- 1.43** Prove that $\sqrt{2}^{\sqrt{2}}$ is either rational or that $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is a rational power of an irrational to an irrational power, and use this to show that an irrational number raised to an irrational power can be rational.
- 1.44** The *floor* function satisfies $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Prove that $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for every integer n , and find $\lfloor x \rfloor + \lfloor -x \rfloor$ for every real x .

CHAPTER 2

LIMITS AND CONTINUITY

LEARNING OBJECTIVES

- Explain the tangent and velocity problems, and estimate limits numerically and graphically.
- State the ϵ - δ definition of a limit and use it to prove limits and nonexistence of limits.
- Prove and apply the limit laws, the Squeeze Theorem, and the fundamental trigonometric limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.
- Compute one-sided limits, infinite limits, and limits at infinity, and find asymptotes.
- Define continuity, classify discontinuities, and apply the Intermediate and Extreme Value Theorems.

Every idea in calculus is built on the limit. The derivative is a limit of difference quotients, the integral a limit of sums, and an infinite series a limit of partial sums. For a century and a half after Newton and Leibniz, calculus worked spectacularly while its foundations remained vague: Bishop Berkeley mocked infinitesimals in 1734 as “the ghosts of departed quantities.” The precise definition of a limit, due to Cauchy and Weierstrass in the nineteenth century, finally put the subject on solid ground. This chapter develops that definition and the tools that make limits computable.

2.1 THE IDEA OF A LIMIT

2.1.1 THE TANGENT PROBLEM

What is the slope of the parabola $y = x^2$ at the point $P = (1, 1)$? A line through P and a second point $Q = (x, x^2)$ on the curve, a *secant line*, has slope

$$m_{PQ} = \frac{x^2 - 1}{x - 1} = x + 1 \quad (x \neq 1).$$

As Q moves toward P , the secant lines approach a limiting line, the tangent, and their slopes approach 2. The formula $x + 1$ cannot be evaluated at $x = 1$ in its original form $\frac{x^2 - 1}{x - 1}$, which is $\frac{0}{0}$ there, but the values approach 2 as x approaches 1. We write

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

2.1.2 THE VELOCITY PROBLEM

A stone dropped from rest falls $s(t) = 4.9t^2$ meters in t seconds. Its average velocity between $t = 2$ and $t = 2 + h$ is

$$\frac{s(2 + h) - s(2)}{h} = \frac{4.9(4 + 4h + h^2) - 19.6}{h} = 19.6 + 4.9h.$$

As $h \rightarrow 0$ the average velocity approaches 19.6 m/s, which we call the *instantaneous velocity* at $t = 2$. The two problems have the same structure: a ratio of changes, evaluated as the change shrinks to zero.

Definition 2.1 (Limit, informal). We write $\lim_{x \rightarrow a} f(x) = L$ if the values $f(x)$ can be made as close to L as we like by taking x sufficiently close to a , but not equal to a .

The condition $x \neq a$ is essential. The limit concerns the behavior of f near a , not at a . The function need not even be defined at a , and if it is, its value there is irrelevant.

Example 2.2 (Estimating a limit numerically). Estimate $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$.

Solution. A table of values:

x	± 1	± 0.5	± 0.1	± 0.01	± 0.001
$f(x)$	0.162278	0.165525	0.166620	0.166666	0.166667

The values suggest the limit $\frac{1}{6}$. Rationalizing confirms it: $\frac{\sqrt{x^2 + 9} - 3}{x^2} =$

$$\frac{(x^2+9)-9}{x^2(\sqrt{x^2+9}+3)} = \frac{1}{\sqrt{x^2+9}+3} \rightarrow \frac{1}{6}.$$

COMMON ERROR (numerical evidence). Tables can mislead. For $x = 10^{-8}$, a calculator evaluating $\frac{\sqrt{x^2+9}-3}{x^2}$ directly returns 0, because $\sqrt{9 + 10^{-16}}$ rounds to exactly 3. This *catastrophic cancellation* afflicts every difference of nearly equal numbers. Numerical estimates suggest limits; only algebra or proof establishes them.

2.1.3 HOW LIMITS FAIL TO EXIST

Example 2.3 (Three failures). (a) *A jump.* The Heaviside function H is 0 for $x < 0$ and 1 for $x \geq 0$. Near 0 it takes values 0 and 1 arbitrarily close to 0, so no single L is approached. $\lim_{x \rightarrow 0} H(x)$ does not exist.

(b) *Unbounded growth.* $\frac{1}{x^2}$ grows without bound as $x \rightarrow 0$. No real L is approached; we will write $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ to describe how it fails.

(c) *Oscillation.* $\sin \frac{1}{x}$ takes every value in $[-1, 1]$ infinitely often in every interval around 0: it equals 1 at $x = \frac{1}{\pi/2 + 2\pi n}$ and -1 at $x = \frac{1}{3\pi/2 + 2\pi n}$, points that accumulate at 0. The limit does not exist.

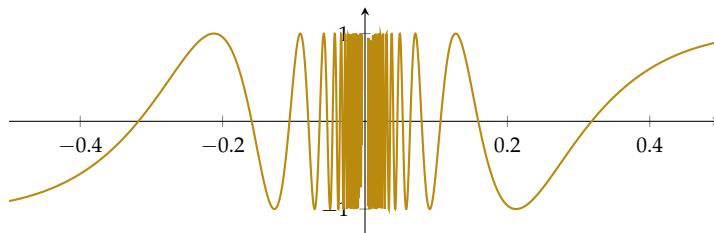


Figure 2.1: $y = \sin(1/x)$ oscillates infinitely often between -1 and 1 near 0 , and has no limit there.

2.2 THE PRECISE DEFINITION OF A LIMIT

The informal phrases “as close as we like” and “sufficiently close” become precise when we name the tolerances. Let $\epsilon > 0$ stand for how close we demand $f(x)$ be to L , and $\delta > 0$ for how close x must be to a to guarantee it.

Definition 2.4 (Limit). Let f be defined on an open interval containing a , except possibly at a . Then $\lim_{x \rightarrow a} f(x) = L$ means: for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

The definition is a challenge and response. A skeptic names a tolerance ϵ , however small. We must produce a δ that works: every x within δ of a (other than a itself, which is what $0 < |x - a|$ excludes) has $f(x)$ within ϵ of L . If we can always answer, the limit is L . Geometrically, for every horizontal band of half-width ϵ about $y = L$, there is a vertical band of half-width δ about $x = a$ inside which the graph stays in the horizontal band.

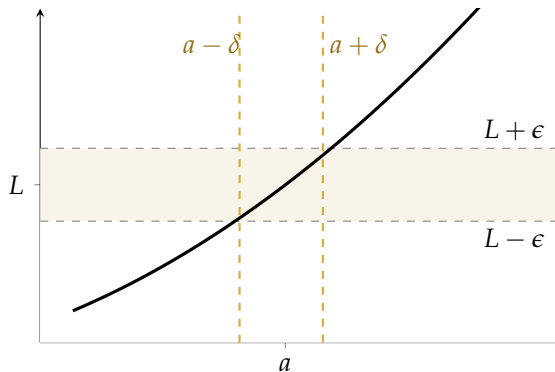


Figure 2.2: The definition of $\lim_{x \rightarrow a} f(x) = L$. Inside the vertical band $|x - a| < \delta$, the graph lies within the horizontal band $|y - L| < \epsilon$.

Example 2.5 (A linear function). Prove that $\lim_{x \rightarrow 2} (3x - 1) = 5$.

Solution. *Scratch work.* We need $|(3x - 1) - 5| < \epsilon$, that is, $3|x - 2| < \epsilon$, that is, $|x - 2| < \epsilon/3$. This suggests $\delta = \epsilon/3$.

Proof. Let $\epsilon > 0$ and set $\delta = \epsilon/3$. If $0 < |x - 2| < \delta$, then $|(3x - 1) - 5| = 3|x - 2| < 3\delta = \epsilon$.

Example 2.6 (A quadratic). Prove that $\lim_{x \rightarrow 3} x^2 = 9$.

Solution. *Scratch work.* $|x^2 - 9| = |x + 3||x - 3|$. The factor $|x - 3|$ can be made small; the factor $|x + 3|$ must be bounded. If we agree to take $\delta \leq 1$, then $|x - 3| < 1$ gives $2 < x < 4$, so $|x + 3| < 7$. Then $|x^2 - 9| < 7|x - 3|$, which is less than ϵ if $|x - 3| < \epsilon/7$.

Proof. Let $\epsilon > 0$ and $\delta = \min(1, \epsilon/7)$. If $0 < |x - 3| < \delta$, then $|x - 3| < 1$, so $|x + 3| < 7$, and $|x - 3| < \epsilon/7$. Hence $|x^2 - 9| = |x + 3||x - 3| < 7 \cdot \frac{\epsilon}{7} = \epsilon$.

The device $\delta = \min(1, \dots)$, which first confines x to a fixed interval to bound the troublesome factor, is the standard technique for nonlinear functions.

Theorem 2.7 (Uniqueness of limits). *If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} f(x) = M$, then $L = M$.*

Proof. Suppose $L \neq M$ and let $\epsilon = \frac{1}{2}|L - M| > 0$. There are δ_1, δ_2 with $|f(x) - L| < \epsilon$ for $0 < |x - a| < \delta_1$ and $|f(x) - M| < \epsilon$ for $0 < |x - a| < \delta_2$. Pick any x with $0 < |x - a| < \min(\delta_1, \delta_2)$. By the triangle inequality, $|L - M| \leq |L - f(x)| + |f(x) - M| < 2\epsilon = |L - M|$, a contradiction. ■

Example 2.8 (Proving a limit does not exist). Prove that $\lim_{x \rightarrow 0} H(x)$ does not exist, where H is the Heaviside function.

Solution. Suppose the limit were L . Take $\epsilon = \frac{1}{2}$. For any $\delta > 0$, the points $x = \frac{\delta}{2}$ and $x = -\frac{\delta}{2}$ satisfy $0 < |x| < \delta$, so we would need $|1 - L| < \frac{1}{2}$ and $|0 - L| < \frac{1}{2}$. Then $1 = |1 - 0| \leq |1 - L| + |L - 0| < 1$, a contradiction. No L works.

2.3 COMPUTING LIMITS

Proving every limit from the definition would be tedious. Instead we prove a few general laws once, and compute with them.

Theorem 2.9 (Limit laws). *Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, and c is a constant. Then*

(a) $\lim_{x \rightarrow a} (f(x) \pm g(x)) = L \pm M$;

(b) $\lim_{x \rightarrow a} cf(x) = cL$;

(c) $\lim_{x \rightarrow a} f(x)g(x) = LM$;

(d) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $M \neq 0$;

(e) $\lim_{x \rightarrow a} (f(x))^n = L^n$ for positive integers n , and $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

(with $L > 0$ if n is even).

Proof. (a) Given $\epsilon > 0$, choose δ_1, δ_2 with $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$ for $0 < |x - a| < \delta_1, \delta_2$ respectively. For $0 < |x - a| < \delta = \min(\delta_1, \delta_2)$, the triangle inequality gives $|(f + g)(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$.

(c) Write $fg - LM = f(g - M) + M(f - L)$. With $\epsilon = 1$ in the definition, there is δ_0 with $|f(x) - L| < 1$, hence $|f(x)| < |L| + 1$, for $0 < |x - a| < \delta_0$. Given $\epsilon > 0$, choose δ_1 with $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$ and δ_2 with $|f(x) - L| < \frac{\epsilon}{2(|M|+1)}$. For $0 < |x - a| < \min(\delta_0, \delta_1, \delta_2)$,

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x)||g(x) - M| + |M||f(x) - L| \\ &< (|L| + 1) \frac{\epsilon}{2(|L| + 1)} + |M| \frac{\epsilon}{2(|M| + 1)} < \epsilon. \end{aligned}$$

(b) follows from (c) with $g = c$. For (d), it suffices by (c) to show $\frac{1}{g} \rightarrow \frac{1}{M}$; this is Exercise 2.31. (e) follows from (c) by induction for powers; the root law is proved in Appendix A. ■

Since $\lim_{x \rightarrow a} c = c$ and $\lim_{x \rightarrow a} x = a$ (take $\delta = \epsilon$), the laws give $\lim_{x \rightarrow a} p(x) = p(a)$ for every polynomial p , and $\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ for rational functions with $q(a) \neq 0$. This is *direct substitution*. When substitution gives $\frac{0}{0}$, the limit may still exist, and algebra is needed to find it.

Example 2.10 (Algebraic techniques). Evaluate (a) $\lim_{x \rightarrow -2} \frac{x^3 + 8}{x^2 - 4}$, (b)

$\lim_{h \rightarrow 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$, (c) $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$.

Solution. (a) Factor: $\frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)} = \frac{x^2-2x+4}{x-2}$ for $x \neq -2$. Since the functions agree near -2 (except at -2 , which does not matter), the limits agree: $\frac{4+4+4}{-4} = -3$.

(b) $\frac{1}{h} \left(\frac{1}{3+h} - \frac{1}{3} \right) = \frac{1}{h} \cdot \frac{3-(3+h)}{3(3+h)} = \frac{-1}{3(3+h)} \rightarrow -\frac{1}{9}$.

(c) Rationalize, or factor $x - 4 = (\sqrt{x} - 2)(\sqrt{x} + 2)$: $\frac{1}{\sqrt{x}+2} \rightarrow \frac{1}{4}$.

2.3.1 THE SQUEEZE THEOREM

Theorem 2.11 (Squeeze Theorem). If $g(x) \leq f(x) \leq h(x)$ for x near a (except possibly at a), and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.

Proof. Given $\epsilon > 0$, choose δ so small that the inequality holds and $L - \epsilon < g(x)$ and $h(x) < L + \epsilon$ for $0 < |x - a| < \delta$. Then $L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon$. ■

Example 2.12 (Squeezing an oscillation). Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

Solution. The limit laws do not apply, since $\lim \sin \frac{1}{x}$ does not exist. But $-1 \leq \sin \frac{1}{x} \leq 1$, so $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$, and both bounds tend to 0. The oscillation persists, but it is crushed by the factor x^2 .

2.3.2 THE FUNDAMENTAL TRIGONOMETRIC LIMIT

Theorem 2.13. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ and $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$.

Proof. Let $0 < \theta < \frac{\pi}{2}$ and consider the unit circle with the point $P = (\cos \theta, \sin \theta)$, the point $A = (1, 0)$, and the point $T = (1, \tan \theta)$ where the ray OP meets the tangent line at A . The triangle OAP lies inside the sector OAP , which lies inside the triangle OAT , so their areas satisfy

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

Dividing by $\frac{1}{2} \sin \theta > 0$ and taking reciprocals, $\cos \theta < \frac{\sin \theta}{\theta} < 1$. Both sides are even functions of θ , so the inequality also holds for $-\frac{\pi}{2} < \theta < 0$. As $\theta \rightarrow 0$, $\cos \theta \rightarrow 1$ (by continuity of cosine, Theorem 2.26), so the Squeeze Theorem gives the first limit. For the second,

$$\frac{1 - \cos \theta}{\theta} = \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} = \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} \rightarrow 1 \cdot \frac{0}{2} = 0. \quad \blacksquare$$

The area of a sector of angle θ in the unit circle is $\frac{1}{2}\theta$ only when θ is in radians. This is the reason radians are the natural unit in calculus: in degrees, the limit would be $\frac{\pi}{180}$.

Example 2.14 (Using the trigonometric limit). Evaluate (a) $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$, (b) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$, (c) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

Solution. (a) $\frac{\sin 5x}{3x} = \frac{5}{3} \cdot \frac{\sin 5x}{5x}$, and $5x \rightarrow 0$, so the limit is $\frac{5}{3}$. (b) $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} \rightarrow 1$. (c) $\frac{1 - \cos x}{x^2} = \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1 + \cos x} \rightarrow \frac{1}{2}$.

2.4 ONE-SIDED AND INFINITE LIMITS

Definition 2.15 (One-sided limits). $\lim_{x \rightarrow a^-} f(x) = L$ (the *left-hand limit*) means: for every $\epsilon > 0$ there is $\delta > 0$ with $|f(x) - L| < \epsilon$ whenever $a - \delta < x < a$. The *right-hand limit* $\lim_{x \rightarrow a^+} f(x)$ is defined with $a < x < a + \delta$.

Theorem 2.16. $\lim_{x \rightarrow a} f(x) = L$ if and only if both one-sided limits exist and equal L .

Proof. If the two-sided limit is L , the same δ serves for each side. Conversely, given ϵ , take δ_- and δ_+ for the two sides; then $\delta = \min(\delta_-, \delta_+)$ works for the two-sided limit. ■

For the Heaviside function, $\lim_{x \rightarrow 0^-} H(x) = 0$ and $\lim_{x \rightarrow 0^+} H(x) = 1$: both exist but differ. For $\sqrt{x}$, only the right-hand limit at 0 makes sense, and $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

Example 2.17 (Absolute value in a quotient). Evaluate $\lim_{x \rightarrow 3} \frac{|x - 3|}{x^2 - 9}$.

Solution. For $x > 3$, $\frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3} \rightarrow \frac{1}{6}$. For $x < 3$, $\frac{-(x-3)}{(x-3)(x+3)} = -\frac{1}{x+3} \rightarrow -\frac{1}{6}$. The one-sided limits differ, so the limit does not exist.

Definition 2.18 (Infinite limits). $\lim_{x \rightarrow a} f(x) = \infty$ means: for every $N > 0$ there is $\delta > 0$ with $f(x) > N$ whenever $0 < |x - a| < \delta$. Similarly for $-\infty$ and for one-sided limits. The line $x = a$ is a *vertical asymptote* of f if at least one one-sided limit at a is $\pm\infty$.

Writing $\lim f = \infty$ does not assert that the limit exists; it describes the particular

way it fails to exist.

Example 2.19 (Vertical asymptotes). Find the vertical asymptotes of $f(x) = \frac{x^2 - x - 2}{x^2 - 4}$ and the behavior near each.

Solution. $f(x) = \frac{(x-2)(x+1)}{(x-2)(x+2)} = \frac{x+1}{x+2}$ for $x \neq \pm 2$. At $x = 2$ the limit is $\frac{3}{4}$: the graph has a *hole*, not an asymptote. At $x = -2$ the numerator tends to -1 while the denominator tends to 0 . For $x \rightarrow -2^+$ the denominator is small and positive, so $f \rightarrow -\infty$; for $x \rightarrow -2^-$ it is small and negative, so $f \rightarrow +\infty$. The only vertical asymptote is $x = -2$.

The vertical asymptotes of $\tan x$ are $x = \frac{\pi}{2} + n\pi$, and that of $\ln x$ is $x = 0$, with $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

2.5 LIMITS AT INFINITY

Definition 2.20 (Limits at infinity). $\lim_{x \rightarrow \infty} f(x) = L$ means: for every $\epsilon > 0$ there is N such that $|f(x) - L| < \epsilon$ whenever $x > N$. Similarly for $x \rightarrow -\infty$. The line $y = L$ is a *horizontal asymptote* if $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$.

The limit laws and the Squeeze Theorem hold for limits at infinity, with the same proofs. The basic limit is $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$ for every $r > 0$: given ϵ , take $N = \epsilon^{-1/r}$.

Example 2.21 (Rational functions at infinity). Evaluate $\lim_{x \rightarrow \infty} \frac{3x^2 - x + 2}{5x^2 + 4x - 1}$ and $\lim_{x \rightarrow \infty} \frac{2x + 1}{x^2 + 3}$.

Solution. Divide numerator and denominator by the highest power of x in the denominator:

$$\frac{3 - \frac{1}{x} + \frac{2}{x^2}}{5 + \frac{4}{x} - \frac{1}{x^2}} \rightarrow \frac{3}{5}, \quad \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} \rightarrow \frac{0}{1} = 0.$$

In general, for p/q with leading terms $a_n x^n$ and $b_m x^m$: the limit is 0 if $n < m$, a_n/b_m if $n = m$, and $\pm\infty$ if $n > m$.

Example 2.22 (A difference of square roots). Evaluate $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - x)$ and $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 3x} + x)$.

Solution. Both have the form $\infty - \infty$, which is indeterminate. Multiply by the conjugate:

$$\sqrt{x^2 + 3x} - x = \frac{3x}{\sqrt{x^2 + 3x} + x} = \frac{3}{\sqrt{1 + 3/x} + 1} \rightarrow \frac{3}{2} \quad (x \rightarrow \infty),$$

where we divided by $x = \sqrt{x^2}$, valid for $x > 0$. For $x \rightarrow -\infty$, $\sqrt{x^2} = |x| = -x$, so $\sqrt{x^2 + 3x} + x = \frac{3x}{\sqrt{x^2 + 3x} - x} = \frac{3x}{-x\sqrt{1 + 3/x} - x} = \frac{-3}{\sqrt{1 + 3/x} + 1} \rightarrow -\frac{3}{2}$.

COMMON ERROR. Writing $\sqrt{x^2} = x$ when x may be negative. The correct identity is $\sqrt{x^2} = |x|$, and it matters every time $x \rightarrow -\infty$.

Example 2.23 (Exponentials and arctangent). $\lim_{x \rightarrow \infty} e^{-x} = 0$ and $\lim_{x \rightarrow -\infty} e^x = 0$, so $y = 0$ is a horizontal asymptote of e^x . $\lim_{x \rightarrow \pm\infty} \arctan x = \pm\frac{\pi}{2}$, so the arctangent has two horizontal asymptotes (Fig. 1.1). And $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$ by squeezing between $\pm\frac{1}{x}$, though the graph crosses the asymptote infinitely often: an asymptote is approached, not necessarily avoided.

2.6 CONTINUITY

Definition 2.24 (Continuity). A function f is *continuous at a* if $\lim_{x \rightarrow a} f(x) = f(a)$. This requires three things: $f(a)$ is defined, the limit exists, and they are equal. In ϵ - δ form: for every $\epsilon > 0$ there is $\delta > 0$ with $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. The function is continuous on an interval if it is continuous at every point (using one-sided limits at endpoints that belong to the interval).

A discontinuity is *removable* if the limit exists but differs from $f(a)$ or $f(a)$ is undefined; a *jump* if the one-sided limits exist but differ; *infinite* if a one-sided limit is infinite; and *essential* (oscillating) otherwise, as for $\sin \frac{1}{x}$ at 0.

Theorem 2.25 (Continuity of combinations). *If f and g are continuous at a , so are $f \pm g$, cf , fg , and f/g (if $g(a) \neq 0$). If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a .*

Proof. The first statement is the limit laws with $L = f(a)$, $M = g(a)$. For the composition, let $\epsilon > 0$. By continuity of f at $b = g(a)$, there is $\eta > 0$ with $|f(y) - f(b)| < \epsilon$ when $|y - b| < \eta$. By continuity of g at a , there is $\delta > 0$ with $|g(x) - b| < \eta$ when $|x - a| < \delta$. Then $|x - a| < \delta$ gives $|f(g(x)) - f(g(a))| < \epsilon$. ■

Theorem 2.26 (Continuity of elementary functions). *Polynomials, rational functions, root functions, exponential and logarithmic functions, the trigonometric functions and their inverses, and the hyperbolic functions are continuous at every point of their domains.*

Proof for sine and cosine. For any h , $|\sin h| \leq |h|$: the chord from $(1,0)$ to $(\cos h, \sin h)$ is shorter than the arc. Hence $\sin h \rightarrow 0$ as $h \rightarrow 0$, and $1 - \cos h = 2 \sin^2 \frac{h}{2} \leq \frac{h^2}{2} \rightarrow 0$. Then $\sin(a+h) = \sin a \cos h + \cos a \sin h \rightarrow \sin a$, and similarly for cosine. The other cases are in Appendix A: polynomials and rational functions follow from the limit laws, and inverses of continuous one-to-one functions on intervals are continuous. ■

Continuity lets us move limits inside continuous functions: $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$ whenever the inner limit b exists and f is continuous at b . For example, $\lim_{x \rightarrow 0} e^{\sin x/x} = e^1$, and $\lim_{x \rightarrow \infty} \arctan\left(\frac{x^2}{x+1}\right) = \frac{\pi}{2}$.

Example 2.27 (Making a piecewise function continuous). For what c is

$$f(x) = \begin{cases} cx^2 + 2x, & x < 2 \\ x^3 - cx, & x \geq 2 \end{cases} \text{ continuous everywhere?}$$

Solution. Each piece is a polynomial, continuous on its own interval. At $x = 2$, the left limit is $4c + 4$ and the right limit and value are $8 - 2c$. Equality requires $4c + 4 = 8 - 2c$, so $c = \frac{2}{3}$.

2.6.1 THE INTERMEDIATE VALUE THEOREM

Theorem 2.28 (Intermediate Value Theorem). *If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there is $c \in [a, b]$ with $f(c) = N$.*

A continuous function on an interval cannot skip values. The proof (Appendix A) uses completeness: c is the supremum of the set of points where $f < N$. Over the rationals the theorem fails: $f(x) = x^2 - 2$ is negative at 1 and positive at 2, but has no rational zero.

Example 2.29 (Locating a root by bisection). Show that $x^3 + x - 1 = 0$ has a root in $(0, 1)$, and locate it to within 0.01.

Solution. $f(x) = x^3 + x - 1$ is continuous with $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so by the IVT there is a root in $(0, 1)$. Since f is increasing (a sum of increasing functions), the root is unique. *Bisection* halves the interval repeatedly, keeping the half on which f changes sign: $f(0.5) = -0.375$, so the root is in $(0.5, 1)$; $f(0.75) = 0.172$, so $(0.5, 0.75)$; $f(0.625) = -0.131$, so $(0.625, 0.75)$; then $(0.625, 0.6875)$, $(0.65625, 0.6875)$, $(0.671875, 0.6875)$, and $(0.6796875, 0.6875)$, of length $0.0078 < 0.01$. The root is $0.6823 \dots$. Each step costs one evaluation and gains one binary digit. Newton's method (Chapter 6) gains far more.

2.6.2 THE EXTREME VALUE THEOREM

Theorem 2.30 (Extreme Value Theorem). *If f is continuous on a closed, bounded interval $[a, b]$, then f attains a maximum and a minimum value: there are $c, d \in [a, b]$ with $f(d) \leq f(x) \leq f(c)$ for all $x \in [a, b]$.*

Each hypothesis is necessary. On the open interval $(0, 1)$, $f(x) = x$ has no maximum. On $[0, \infty)$, $f(x) = x$ has none either. And on $[-1, 1]$, the discontinuous function equal to $\frac{1}{x}$ for $x \neq 0$ and 0 at 0 is unbounded. The proof is in Appendix A. The theorem guarantees that the optimization problems of Chapter 6 have solutions.

LOOKING AHEAD. Continuity on a set is a property of each point separately: δ may depend on both ϵ and the point. When a single δ works for every point of the set, f is *uniformly continuous*. On a closed bounded interval, continuity implies uniform continuity, and that is exactly what is needed to

prove that continuous functions are integrable (Appendix A).

2.7 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 2.1 An ϵ - δ proof for a square root

Prove from the definition that $\lim_{x \rightarrow 4} \sqrt{x} = 2$.

Solution. *Scratch work.* Rationalize: $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2} \leq \frac{|x-4|}{2}$, since $\sqrt{x} \geq 0$. So $|x - 4| < 2\epsilon$ suffices. We also need $x \geq 0$ for $\sqrt{x}$ to be defined, which holds if $|x - 4| < 4$.

Proof. Let $\epsilon > 0$ and $\delta = \min(4, 2\epsilon)$. If $0 < |x - 4| < \delta$, then $x > 0$, and $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2} \leq \frac{|x-4|}{2} < \frac{2\epsilon}{2} = \epsilon$.

The rationalizing identity, which converted a difference of square roots into a quotient with a bounded denominator, is the same algebra used to evaluate limits like Example 2.2. Proof and computation use the same tools.

— Problem 2.2 An ϵ - δ proof for a reciprocal

Prove that $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$.

Solution. *Scratch work.* $|\frac{1}{x} - \frac{1}{2}| = \frac{|x-2|}{2|x|}$. We must keep $|x|$ away from 0. If $|x - 2| < 1$, then $1 < x < 3$, so $\frac{1}{2|x|} < \frac{1}{2}$ and $|\frac{1}{x} - \frac{1}{2}| < \frac{|x-2|}{2}$.

Proof. Given $\epsilon > 0$, let $\delta = \min(1, 2\epsilon)$. If $0 < |x - 2| < \delta$, then $x > 1$ and $|\frac{1}{x} - \frac{1}{2}| = \frac{|x-2|}{2x} < \frac{|x-2|}{2} < \epsilon$.

The general fact behind the quotient law is the same: if $g \rightarrow M \neq 0$, then eventually $|g(x)| > \frac{|M|}{2}$, which keeps $\frac{1}{g}$ under control.

— Problem 2.3 A limit that needs the squeeze

Evaluate $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x + \cos x}$ and $\lim_{x \rightarrow 0} x \left\lfloor \frac{1}{x} \right\rfloor$.

Solution. For the first, divide by x : $\frac{1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{x}}$. Since $|\sin x| \leq 1$, $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for $x > 0$, so $\frac{\sin x}{x} \rightarrow 0$ by squeezing; likewise $\frac{\cos x}{x} \rightarrow 0$. The limit is $\frac{1+0}{1+0} = 1$.

For the second, the floor satisfies $\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}$. For $x > 0$, multiply by x :

$1 - x < x \lfloor \frac{1}{x} \rfloor \leq 1$. For $x < 0$, multiplying reverses the inequalities: $1 \leq x \lfloor \frac{1}{x} \rfloor < 1 - x$. In both cases the bounds tend to 1, so the limit is 1. The function itself jumps infinitely often near 0, but the jumps shrink.

— **Problem 2.4** Asymptotes of a rational function

Find all asymptotes, including any slant asymptote, of $f(x) = \frac{2x^2 - 3x - 2}{x - 1}$.

Solution. The denominator vanishes at $x = 1$, where the numerator is $2 - 3 - 2 = -3 \neq 0$. So $x = 1$ is a vertical asymptote: as $x \rightarrow 1^+$, $f \rightarrow -\infty$, and as $x \rightarrow 1^-$, $f \rightarrow +\infty$. There is no horizontal asymptote, since the numerator has higher degree. Long division gives

$$f(x) = 2x - 1 - \frac{3}{x - 1}.$$

Since $\frac{3}{x-1} \rightarrow 0$ as $x \rightarrow \pm\infty$, $f(x) - (2x - 1) \rightarrow 0$: the line $y = 2x - 1$ is a *slant asymptote*. The graph lies below the line for $x > 1$ and above it for $x < 1$. A rational function has a slant asymptote exactly when the degree of the numerator is one more than that of the denominator.

— **Problem 2.5** Three real roots

Show that $p(x) = x^5 - 3x + 1$ has at least three real roots, and locate each in an interval of length 1.

Solution. Evaluate at integers: $p(-2) = -32 + 6 + 1 = -25$, $p(-1) = -1 + 3 + 1 = 3$, $p(0) = 1$, $p(1) = -1$, $p(2) = 32 - 6 + 1 = 27$. The sign changes on $[-2, -1]$, $[0, 1]$, and $[1, 2]$. By the IVT there is a root in each, and the intervals overlap only at the endpoint 1, where $p \neq 0$, so the roots are distinct. (A fifth-degree polynomial has at most five roots by Theorem 1.14. The derivative test of Chapter 5 will show there are exactly three: $p'(x) = 5x^4 - 3$ vanishes at only two points, so p changes direction only twice.) Numerically the roots are -1.3888 , 0.3347 , and 1.2146 .

— **Problem 2.6** Continuous at exactly one point

Let $D(x) = 1$ if x is rational and 0 if x is irrational. Prove that D is discontinuous at every point, and that $f(x) = xD(x)$ is continuous at 0 and nowhere else.

Solution. Every interval contains both rationals and irrationals (Exercise 1.32 and its irrational analog, obtained by adding $\sqrt{2}$ to a rational). So for any a and any $\delta > 0$, D takes both values 0 and 1 in $(a - \delta, a + \delta)$. With $\epsilon = \frac{1}{2}$, no L can

satisfy $|D(x) - L| < \frac{1}{2}$ for both values. Hence $\lim_{x \rightarrow a} D(x)$ does not exist, and D is discontinuous everywhere.

For $f = xD$: at 0, $|f(x) - f(0)| = |x|D(x) \leq |x|$, so $\delta = \epsilon$ works, and f is continuous at 0. At $a \neq 0$, f takes values near a (at rationals) and the value 0 (at irrationals) arbitrarily close to a . With $\epsilon = \frac{|a|}{2}$, the limit cannot exist. So f is continuous only at 0. Pathological functions like these show that continuity is a pointwise condition, and that “a graph you can draw without lifting the pencil” is an intuition, not a definition.

— Problem 2.7 Fixed points

Prove that every continuous function $f : [0, 1] \rightarrow [0, 1]$ has a *fixed point*: a c with $f(c) = c$. Show by example that the conclusion can fail for a continuous $f : (0, 1) \rightarrow (0, 1)$.

Solution. Let $g(x) = f(x) - x$, continuous on $[0, 1]$. Since $0 \leq f \leq 1$, $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. If either is zero we are done. Otherwise $g(0) > 0 > g(1)$, and the IVT gives $c \in (0, 1)$ with $g(c) = 0$, that is, $f(c) = c$.

On the open interval, $f(x) = \frac{x}{2}$ maps $(0, 1)$ into $(0, 1)$ continuously, but $\frac{x}{2} = x$ only at $x = 0$, which is not in the interval. This is the one-dimensional case of the *Brouwer fixed-point theorem*, which says the same for continuous maps of a closed disk or ball to itself. It underlies existence proofs throughout mathematics and economics, including the existence of Nash equilibria.

EXERCISES

A. COMPUTATION

2.1 Estimate $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$ from a table with $x = \pm 0.1, \pm 0.01, \pm 0.001$.

2.2 The position of a car is $s(t) = t^3 - 2t$ meters. Find its average velocity on $[1, 1 + h]$ and its instantaneous velocity at $t = 1$.

2.3 Prove from the definition: $\lim_{x \rightarrow -1} (4 - 5x) = 9$.

2.4 Prove from the definition: $\lim_{x \rightarrow 2} (x^2 + x) = 6$.

2.5 Prove from the definition: $\lim_{x \rightarrow 0} x^3 = 0$.

2.6 Find δ such that $|x - 1| < \delta$ implies $|x^2 - 1| < 0.01$.

2.7 Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 2x - 3}$.

2.8 Evaluate $\lim_{t \rightarrow 0} \frac{\sqrt{4+t} - 2}{t}$.

2.9 Evaluate $\lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h}$.

2.10 Evaluate $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^3 - 1}$.

2.11 Evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 7x}$.

2.12 Evaluate $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$ and $\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos x}$.

2.13 Evaluate $\lim_{x \rightarrow 0} x \cot x$.

2.14 Evaluate $\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x}$.

2.15 Evaluate $\lim_{x \rightarrow -4^+} \frac{x+3}{x+4}$ and $\lim_{x \rightarrow -4^-} \frac{x+3}{x+4}$.

2.16 Evaluate $\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2}$ and $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$.

2.17 Find the vertical and horizontal asymptotes of $f(x) = \frac{2x^2 + 1}{x^2 - x - 2}$.

2.18 Find the vertical asymptotes of $f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$, and identify any removable discontinuity.

2.19 Evaluate $\lim_{x \rightarrow \infty} \frac{4x^3 - x}{2x^3 + 5x^2}$ and $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 1}}{2x - 3}$.

2.20 Evaluate $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$.

2.21 Evaluate $\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and $\lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

2.22 Evaluate $\lim_{x \rightarrow \infty} \arctan(x^2 - x)$ and $\lim_{x \rightarrow 0^+} \ln(\sin x)$.

2.23 Find the slant asymptote of $f(x) = \frac{x^3 + 1}{x^2 + 1}$.

2.24 Classify the discontinuities of $f(x) = \frac{x^2 - 1}{|x - 1|}$.

2.25 Find a and b so that $f(x) = \begin{cases} x + 1, & x < 1 \\ ax^2 + b, & 1 \leq x < 2 \\ 3x, & x \geq 2 \end{cases}$ is continuous everywhere.

2.26 Where is $f(x) = \frac{\ln x}{\sqrt{4 - x^2}}$ continuous?

2.27 Show that $\cos x = x$ has a solution in $(0, 1)$, and find it to two decimal places by bisection.

2.28 Show that $e^x = 3 - x$ has exactly one real solution.

2.29 Find $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x}$ and $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$.

2.30 Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$.

B. THEORY AND PROOF

2.31 Prove that if $\lim_{x \rightarrow a} g(x) = M \neq 0$, then $\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$.

2.32 Prove that if $\lim_{x \rightarrow a} f(x) = L > 0$, then $f(x) > 0$ for all $x \neq a$ sufficiently close to a .

2.33 Prove that if $f(x) \leq g(x)$ near a and both limits exist, then $\lim_{x \rightarrow a} f \leq \lim_{x \rightarrow a} g$.

2.34 Prove that $\lim_{x \rightarrow a} f(x) = 0$ if and only if $\lim_{x \rightarrow a} |f(x)| = 0$. Is the analogous statement true with 0 replaced by $L \neq 0$?

2.35 Prove that if f is bounded near a and $\lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} f(x)g(x) = 0$.

2.36 Prove from the definition that $\lim_{x \rightarrow \infty} \frac{x}{x+1} = 1$.

2.37 Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ from the definition.

2.38 Give an example of functions f, g with no limits at 0 such that $f + g$ and fg both have limits at 0.

2.39 Prove that a polynomial of odd degree has at least one real root.

2.40 Prove that if f is continuous on $[a, b]$ and $f(x) \neq 0$ for all x , then f has the same sign throughout $[a, b]$.

C. CHALLENGE

2.41 Prove that $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ for every $a > 0$ from the definition.

2.42 The *Thomae function* is $T(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ in lowest terms with $q > 0$, and $T(x) = 0$ if x is irrational. Prove that T is continuous at every irrational and discontinuous at every rational.

2.43 A hiker starts up a mountain at 7:00 a.m. and reaches the top at 7:00 p.m. The next day she starts down at 7:00 a.m. by the same path and reaches the bottom at 7:00 p.m. Prove that there is a point on the path that she passes at the same time on both days.

2.44 Prove that there is no continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ that takes every value exactly twice.

2.45 Prove that if f is continuous on $\mathbb{R}$ and $f(x + y) = f(x) + f(y)$ for all x, y , then $f(x) = cx$ for some constant c . (First show it for integers, then rationals, then use continuity.)

CHAPTER 3

THE DERIVATIVE

LEARNING OBJECTIVES

- Define the derivative at a point as a limit, and interpret it as a slope and as an instantaneous rate of change.
- Prove that differentiability implies continuity, and recognize corners, cusps, and vertical tangents.
- Derive and apply the power, sum, product, and quotient rules.
- Define the number e by the slope of an exponential, and differentiate e^x and the trigonometric functions.
- Use derivatives to describe velocity, acceleration, and rates of change in the sciences.

The tangent and velocity problems of Chapter 2 led to the same limit: a ratio of changes, as the change shrinks to zero. That limit is the derivative, the central object of differential calculus. This chapter defines it, computes it for the basic functions, and derives the rules that compute it for their sums, products, and quotients. The chain rule, which handles compositions, is the subject of Chapter 4.

3.1 THE DERIVATIVE AT A POINT

Definition 3.1 (Derivative at a point). The *derivative* of f at a is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a},$$

if the limit exists. Then f is *differentiable at a* . The two forms are related by $x = a + h$.

The quotient $\frac{f(a+h)-f(a)}{h}$ is the *difference quotient*: the slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$, and the average rate of change of f over $[a, a+h]$. Its limit is the slope of the *tangent line* to the graph at $(a, f(a))$, whose equation is

$$y = f(a) + f'(a)(x - a),$$

and the *instantaneous rate of change* of f at a .

Example 3.2 (Three derivatives from the definition). Find $f'(a)$ for (a) $f(x) = x^2$, (b) $f(x) = \frac{1}{x}$, (c) $f(x) = \sqrt{x}$ ($a > 0$).

Solution. (a) $\frac{(a+h)^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h \rightarrow 2a$.

(b) $\frac{1}{h} \left(\frac{1}{a+h} - \frac{1}{a} \right) = \frac{1}{h} \cdot \frac{-h}{a(a+h)} = -\frac{1}{a(a+h)} \rightarrow -\frac{1}{a^2}$.

(c) $\frac{\sqrt{a+h} - \sqrt{a}}{h} = \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})} = \frac{1}{\sqrt{a+h} + \sqrt{a}} \rightarrow \frac{1}{2\sqrt{a}}$.

In each case the difference quotient was $\frac{0}{0}$ at $h = 0$, and algebra removed the factor h before the limit was taken. That is the whole art of computing derivatives from the definition.

Example 3.3 (A tangent line). Find the tangent line to $y = \frac{1}{x}$ at $(2, \frac{1}{2})$, and the points where it meets the axes.

Solution. By (b), the slope is $-\frac{1}{4}$. The tangent line is $y = \frac{1}{2} - \frac{1}{4}(x - 2) = 1 - \frac{x}{4}$. It meets the axes at $(0, 1)$ and $(4, 0)$. Notice that the point of tangency $(2, \frac{1}{2})$ is the midpoint of the segment between the intercepts. That is true for every tangent to the hyperbola $xy = 1$ (Exercise 3.35).

3.1.1 RATES OF CHANGE

If $y = f(x)$, the change $\Delta y = f(x + \Delta x) - f(x)$ divided by Δx is the average rate of change, and $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ is the instantaneous rate. The units of f' are the units of y per unit of x . If $s(t)$ is position in meters and t is time in seconds, $s'(t)$ is velocity in m/s. If $C(q)$ is the cost in dollars of producing q items, $C'(q)$, the *marginal cost*, is in dollars per item, and approximates the cost of producing one more item.

3.2 THE DERIVATIVE AS A FUNCTION

Letting the point vary, the derivative becomes a new function f' , defined wherever the limit exists:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Other notations are $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f(x)$, $Df(x)$, and $\dot{y}$ (for derivatives with respect to time). *Leibniz notation* $\frac{dy}{dx}$ recalls the ratio $\frac{\Delta y}{\Delta x}$ it is the limit of. It is not a fraction, but it often behaves like one, and that is its great virtue. The value at a point is written $\left. \frac{dy}{dx} \right|_{x=a}$ or $f'(a)$.

Theorem 3.4 (Differentiability implies continuity). *If f is differentiable at a , then f is continuous at a .*

Proof. For $x \neq a$, $f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$. As $x \rightarrow a$, the first factor tends to $f'(a)$ and the second to 0, so by the product law $f(x) - f(a) \rightarrow f'(a) \cdot 0 = 0$. ■

The converse is false. A continuous function can fail to be differentiable in three ways:

- a *corner*, where the one-sided limits of the difference quotient differ: $|x|$ at 0, with one-sided slopes ± 1 ;
- a *vertical tangent*, where the difference quotient tends to $\pm\infty$: $\sqrt[3]{x}$ at 0, where $\frac{x^{1/3}}{x} = x^{-2/3} \rightarrow \infty$;
- a *cusp*, a vertical tangent with the quotient tending to $+\infty$ on one side and $-\infty$ on the other: $x^{2/3}$ at 0.

Worse is possible. Weierstrass gave in 1872 a function, $\sum_{n \geq 0} a^n \cos(b^n \pi x)$ with $0 < a < 1$, b an odd integer, and $ab > 1 + \frac{3\pi}{2}$, that is continuous everywhere and differentiable nowhere: its graph has corners at every scale.

3.2.1 HIGHER DERIVATIVES

The derivative of f' is the *second derivative* $f'' = \frac{d^2y}{dx^2}$, and so on: f''' , $f^{(4)}$, ..., $f^{(n)} = \frac{d^ny}{dx^n}$. If $s(t)$ is position, $v = s'$ is velocity and $a = v' = s''$ is acceleration. The third derivative, the rate of change of acceleration, is called *jerk*, and engineers limit it in the design of elevators and roller coasters.

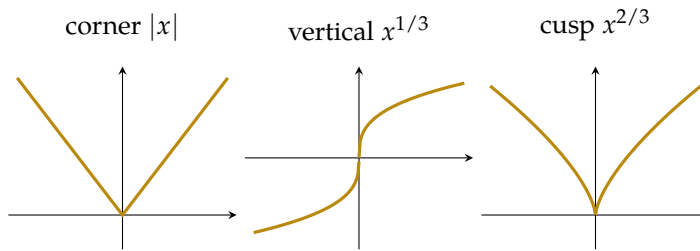


Figure 3.1: Three continuous functions that are not differentiable at 0.

Example 3.5 (Reading a derivative from a graph). The graph of f' can be sketched from that of f : f' is positive where f rises, negative where f falls, and zero at the peaks and troughs of f , where the tangent is horizontal. Steep stretches of f correspond to large $|f'|$. For $f(x) = x^3 - 3x$, the peaks and troughs are at $x = \pm 1$; f rises on $(-\infty, -1)$ and $(1, \infty)$ and falls on $(-1, 1)$; and the derivative, $3x^2 - 3$, is a parabola with exactly that sign pattern.

3.3 DIFFERENTIATION RULES

Theorem 3.6 (Constant, power, sum, and constant multiple rules). For constants c and positive integers n , and differentiable f, g :

$$\frac{d}{dx}c = 0, \quad \frac{d}{dx}x^n = nx^{n-1}, \quad (f \pm g)' = f' \pm g', \quad (cf)' = cf'.$$

Proof. For the power rule, the binomial theorem gives $(x + h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n$. So

$$\frac{(x + h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1} \rightarrow nx^{n-1},$$

since every term after the first contains a factor of h . The sum and constant multiple rules follow from the limit laws. ■

The power rule $\frac{d}{dx}x^r = rx^{r-1}$ holds for every real exponent r , wherever x^r is defined. We have seen $r = -1$ and $r = \frac{1}{2}$ in Example 3.2. Negative integers follow from the quotient rule below, rational exponents from implicit differentiation (Chapter 4), and the general case from logarithmic differentiation.

Theorem 3.7 (Product and quotient rules). If f and g are differentiable at x , then so are fg and, where $g(x) \neq 0$, f/g , and

$$(fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Proof. For the product, add and subtract $f(x+h)g(x)$ in the numerator:

$$\begin{aligned} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= f(x+h) \frac{g(x+h) - g(x)}{h} \\ &\quad + g(x) \frac{f(x+h) - f(x)}{h}. \end{aligned}$$

As $h \rightarrow 0$, $f(x+h) \rightarrow f(x)$ by Theorem 3.4, and the quotients tend to $g'(x)$ and $f'(x)$. For the quotient, first $\frac{1}{h} \left(\frac{1}{g(x+h)} - \frac{1}{g(x)} \right) = -\frac{g(x+h)-g(x)}{h} \cdot \frac{1}{g(x+h)g(x)} \rightarrow -\frac{g'}{g^2}$, and then the product rule applied to $f \cdot \frac{1}{g}$ gives $\frac{f'}{g} - \frac{fg'}{g^2}$. ■

COMMON ERROR. The derivative of a product is not the product of the derivatives. For $f = g = x$, $(fg)' = (x^2)' = 2x$, but $f'g' = 1$. Leibniz himself first conjectured $(fg)' = f'g'$ in his notebooks, and corrected it within days.

Example 3.8 (Using the rules). Differentiate (a) $y = (x^2 + 1)(x^3 - 2x)$, (b) $y = \frac{x^2 - 1}{x^2 + 1}$, (c) $y = \frac{3}{x^4} - 2\sqrt[3]{x}$.

Solution. (a) $(2x)(x^3 - 2x) + (x^2 + 1)(3x^2 - 2) = 5x^4 - 3x^2 - 2$. (Expanding first, $x^5 - x^3 - 2x$, gives the same.)

(b) $\frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$

(c) Write $y = 3x^{-4} - 2x^{1/3}$. Then $y' = -12x^{-5} - \frac{2}{3}x^{-2/3}$.

Example 3.9 (Repeated products: Leibniz's rule). Show that $(fg)'' = f''g + 2f'g' + fg''$, and in general $(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$.

Solution. Differentiate $(fg)' = f'g + fg'$ once more with the product rule: $f''g + f'g' + f'g' + fg''$. The general formula follows by induction, in exactly the way the binomial theorem does, since differentiating each term $f^{(k)}g^{(n-k)}$ produces $f^{(k+1)}g^{(n-k)} + f^{(k)}g^{(n-k+1)}$, and Pascal's rule $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ collects the coefficients.

3.4 THE EXPONENTIAL FUNCTION AND THE NUMBER e

For $f(x) = b^x$ with $b > 0$,

$$f'(x) = \lim_{h \rightarrow 0} \frac{b^{x+h} - b^x}{h} = b^x \lim_{h \rightarrow 0} \frac{b^h - 1}{h} = b^x f'(0).$$

The derivative of an exponential is proportional to the exponential itself, and the constant of proportionality is the slope $f'(0)$ of its graph at $x = 0$. Numerically, $\frac{2^h - 1}{h} \rightarrow 0.6931 \dots$ and $\frac{3^h - 1}{h} \rightarrow 1.0986 \dots$. The slope increases continuously with b , from below 1 at $b = 2$ to above 1 at $b = 3$, so there is exactly one base for which it equals 1.

Definition 3.10 (The number e). e is the unique positive number with $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$. Numerically $e = 2.718\,281\,828\,459 \dots$

Theorem 3.11 (Derivative of the natural exponential). $\frac{d}{dx}e^x = e^x$.

The function e^x is its own derivative. That property, rather than any numerical accident, is why e is the natural base: it appears whenever a quantity grows at a rate proportional to its size, from compound interest to radioactive decay to population growth (Chapter 10).

The argument above assumed that the limit $\frac{b^h - 1}{h}$ exists and depends continuously on b . Both facts are proved in Chapter 7, where $\ln x$ is defined as an integral and e^x as its inverse. There we will also show that the slope for base b is exactly $\ln b$, which explains the numbers $0.6931 = \ln 2$ and $1.0986 = \ln 3$, and that $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$.

Example 3.12 (Exponential products and quotients). Differentiate $y = x^2 e^x$ and $y = \frac{e^x}{1+x}$.

Solution. $(x^2 e^x)' = 2x e^x + x^2 e^x = x(x+2)e^x$. And $\left(\frac{e^x}{1+x}\right)' = \frac{e^x(1+x) - e^x}{(1+x)^2} = \frac{x e^x}{(1+x)^2}$. The derivative is zero only at $x = 0$, where the graph of $\frac{e^x}{1+x}$ has a horizontal tangent.

3.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Theorem 3.13.

$$\begin{aligned}\frac{d}{dx} \sin x &= \cos x, & \frac{d}{dx} \cos x &= -\sin x, & \frac{d}{dx} \tan x &= \sec^2 x, \\ \frac{d}{dx} \csc x &= -\csc x \cot x, & \frac{d}{dx} \sec x &= \sec x \tan x, & \frac{d}{dx} \cot x &= -\csc^2 x.\end{aligned}$$

Proof. By the addition formula and Theorem 2.13,

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \rightarrow \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Similarly $\frac{\cos(x+h) - \cos x}{h} = \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \rightarrow -\sin x$. The other four follow from the quotient rule; for example, $(\tan x)' = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$. ■

Example 3.14 (Periodic patterns). Find $\frac{d^{99}}{dx^{99}} \sin x$ and $\frac{d^4}{dx^4} (x \sin x)$.

Solution. The derivatives of $\sin x$ cycle with period 4: $\cos x, -\sin x, -\cos x, \sin x$. Since $99 = 4 \cdot 24 + 3$, the 99th derivative equals the third, $-\cos x$. For $x \sin x$, Leibniz's rule (Example 3.9) with $f = x$, whose derivatives vanish after the first, gives $(x \sin x)^{(4)} = x \sin^{(4)} x + 4 \sin^{(3)} x = x \sin x - 4 \cos x$.

Example 3.15 (Simple harmonic motion). A mass on a spring oscillates with position $x(t) = 4 \cos 2t$ cm. Find its velocity and acceleration, and the relation between acceleration and position.

Solution. $v = x' = -8 \sin 2t$ and $a = v' = -16 \cos 2t = -4x$. (We used $\frac{d}{dt} \cos 2t = -2 \sin 2t$, an instance of the chain rule of Chapter 4: verify it directly with the addition formula.) The acceleration is proportional to the displacement and opposite in sign, the defining property of simple harmonic motion. The speed is greatest, 8 cm/s, at the equilibrium $x = 0$, and the acceleration is greatest at the turning points $x = \pm 4$.

3.6 RATES OF CHANGE IN THE SCIENCES

3.6.1 MOTION ALONG A LINE

If $s(t)$ is the position of a particle, then $v(t) = s'(t)$ is its velocity, $|v(t)|$ its speed, and $a(t) = v'(t) = s''(t)$ its acceleration. The particle speeds up when v and a have the same sign and slows down when they have opposite signs.

Example 3.16 (A particle reversing direction). A particle moves with $s(t) = t^3 - 6t^2 + 9t$ meters for $t \geq 0$. When is it at rest? When is it moving forward? Find the total distance traveled during the first 5 seconds, and the intervals on which it speeds up.

Solution. $v(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$ and $a(t) = 6t - 12 = 6(t - 2)$. It is at rest at $t = 1$ and $t = 3$. It moves forward ($v > 0$) on $[0, 1)$ and $(3, \infty)$, and backward on $(1, 3)$. Positions: $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, $s(5) = 20$. The distance traveled is $|4 - 0| + |0 - 4| + |20 - 0| = 28$ m, though the displacement is only 20 m. It speeds up where v and a have the same sign: on $(1, 2)$, where both are negative, and on $(3, \infty)$, where both are positive. It slows down on $(0, 1)$ and $(2, 3)$.

3.6.2 OTHER SCIENCES

- Example 3.17** (Rates in chemistry, biology, and economics). (a) *Reaction rate.* If $[A](t)$ is the concentration of a reactant, $-\frac{d[A]}{dt}$ is the rate of the reaction.
- (b) *Population growth.* If $P(t)$ is a population, $P'(t)$ is the growth rate, and $P'(t)/P(t)$ is the *relative* or per capita growth rate.
- (c) *Electric current.* If $Q(t)$ is the charge that has passed a point in a wire, $I = \frac{dQ}{dt}$ is the current, in coulombs per second, or amperes.
- (d) *Linear density.* If $m(x)$ is the mass of the first x meters of a rod, $\rho = \frac{dm}{dx}$ is its density in kg/m.
- (e) *Marginal cost.* If $C(q) = 500 + 3q + 0.01q^2$ dollars, then $C'(q) = 3 + 0.02q$, and $C'(100) = 5$ dollars per item, close to the exact cost of the 101st item, $C(101) - C(100) = 5.01$.

LOOKING AHEAD. Every law of motion in physics is a statement about derivatives. Newton's second law says force equals mass times the second

derivative of position. In Volume IV, the same idea, applied to a wave instead of a particle, produces the Schrödinger equation, which governs every atom.

3.7 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 3.1 Joining two pieces smoothly

Find a and b so that $f(x) = \begin{cases} x^2, & x \leq 1 \\ ax + b, & x > 1 \end{cases}$ is differentiable everywhere.

Solution. Differentiability at 1 requires continuity there (Theorem 3.4), so the left value 1 must equal the right limit $a + b$. It also requires the one-sided difference quotients to agree. From the left, $\lim_{h \rightarrow 0^-} \frac{(1+h)^2 - 1}{h} = 2$. From the right, using $a + b = 1$, $\lim_{h \rightarrow 0^+} \frac{a(1+h) + b - 1}{h} = \lim_{h \rightarrow 0^+} \frac{ah}{h} = a$. So $a = 2$ and $b = -1$: the line $y = 2x - 1$ is the tangent to the parabola at $(1, 1)$, as it must be. If we had matched only the derivative formulas, $2x$ and a , without imposing continuity, we would have found $a = 2$ but no condition on b ; the resulting function would have a jump and no derivative at 1.

— Problem 3.2 Tangent lines through an outside point

Find all lines through $(1, -3)$ that are tangent to the parabola $y = x^2$.

Solution. The point $(1, -3)$ is below the parabola, so it is not a point of tangency. Let the tangency be at (t, t^2) . The tangent there is $y = t^2 + 2t(x - t) = 2tx - t^2$. It passes through $(1, -3)$ when $-3 = 2t - t^2$, that is, $t^2 - 2t - 3 = (t - 3)(t + 1) = 0$. So $t = 3$ or $t = -1$, and the lines are

$$y = 6x - 9 \quad \text{and} \quad y = -2x - 1.$$

The method, parametrizing the unknown point of tangency and imposing the condition, works for any curve. Every point below a parabola lies on exactly two tangents, every point above on none, and every point on the parabola on exactly one.

— Problem 3.3 A common tangent

Find the lines tangent to both $y = x^2$ and $y = -x^2 + 2x - 5$.

Solution. The tangent to $y = x^2$ at $x = a$ is $y = 2ax - a^2$. The tangent to $g(x) = -x^2 + 2x - 5$ at $x = b$ has slope $g'(b) = 2 - 2b$ and passes through $(b, g(b))$, so it is $y = (2 - 2b)x + b^2 - 5$ (the intercept is $g(b) - g'(b)b = -b^2 + 2b - 5 - (2 - 2b)b = b^2 - 5$). The lines coincide when slopes and intercepts agree:

$$2a = 2 - 2b, \quad -a^2 = b^2 - 5.$$

Then $a = 1 - b$, and $-(1 - b)^2 = b^2 - 5$ simplifies to $b^2 - b - 2 = 0$, so $b = 2$ or $b = -1$, with $a = -1$ or $a = 2$. The common tangents are $y = -2x - 1$ (touching the first parabola at $x = -1$ and the second at $x = 2$) and $y = 4x - 4$ (touching at $x = 2$ and $x = -1$).

— **Problem 3.4** The derivative of a cube root, from the definition

Prove that $\frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3}$ for $x \neq 0$, and that $x^{1/3}$ is not differentiable at 0.

Solution. Let $u = (x + h)^{1/3}$ and $v = x^{1/3}$. The factorization $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$ plays the role that rationalizing played for square roots:

$$\frac{u - v}{h} = \frac{u - v}{u^3 - v^3} = \frac{1}{u^2 + uv + v^2}.$$

As $h \rightarrow 0$, $u \rightarrow v$ by continuity of the cube root, and the denominator tends to $3v^2 = 3x^{2/3}$, which is nonzero for $x \neq 0$. So the derivative is $\frac{1}{3x^{2/3}}$. At $x = 0$, the difference quotient is $\frac{h^{1/3}}{h} = h^{-2/3} \rightarrow +\infty$: a vertical tangent (Fig. 3.1).

— **Problem 3.5** Where is $|x^2 - 4|$ differentiable?

Find the derivative of $f(x) = |x^2 - 4|$ where it exists, and identify the points where it does not.

Solution. $f(x) = x^2 - 4$ where $|x| \geq 2$ and $f(x) = 4 - x^2$ where $|x| < 2$. On the open sets $|x| > 2$ and $|x| < 2$, f agrees with a polynomial, so $f'(x) = 2x$ for $|x| > 2$ and $f'(x) = -2x$ for $|x| < 2$. At $x = 2$, $f(2) = 0$ and the one-sided difference quotients are

$$\lim_{h \rightarrow 0^+} \frac{(2+h)^2 - 4}{h} = 4, \quad \lim_{h \rightarrow 0^-} \frac{4 - (2+h)^2}{h} = \lim_{h \rightarrow 0^-} (-4 - h) = -4.$$

They differ, so f has a corner at 2; by symmetry also at -2 . In general $|g|$ has a corner wherever g crosses zero with nonzero slope, and is differentiable where g touches zero with zero slope: $|x^3|$ is differentiable at 0.

— **Problem 3.6** Pinning down e

Using only the definition of e , the fact that $\frac{b^h-1}{h}$ increases with b , and a calculator, show that $2.718 < e < 2.719$.

Solution. Approximate $m(b) = \lim_{h \rightarrow 0} \frac{b^h-1}{h}$ with $h = 10^{-7}$, small enough that the error is about 10^{-7} but large enough to avoid the cancellation warned of in Chapter 2:

b	2.7	2.71	2.718	2.719	2.72
$m(b)$	0.99325	0.99695	0.99990	1.00026	1.00063

Since $m(2.718) < 1 < m(2.719)$ with margins far larger than the approximation error, and m increases, the base with $m(b) = 1$ lies between them. This is bisection (Example 2.29) applied to the function $m(b) - 1$. With the logarithm of Chapter 7, $m(b) = \ln b$, and the table is a table of natural logarithms.

— **Problem 3.7** A projectile

A ball is thrown upward from a height of 1.5 m at 20 m/s, so its height is $h(t) = 1.5 + 20t - 4.9t^2$ meters. Find its maximum height, the time it hits the ground, and its speed at impact.

Solution. $v(t) = h'(t) = 20 - 9.8t$, which is zero at $t = \frac{20}{9.8} \approx 2.041$ s. There $h = 1.5 + \frac{20^2}{2(9.8)} \approx 21.91$ m: the maximum, since the ball rises before and falls after. The ball lands when $h(t) = 0$:

$$t = \frac{20 + \sqrt{400 + 4(4.9)(1.5)}}{9.8} = \frac{100 + \sqrt{10735}}{49} \approx 4.155 \text{ s,}$$

taking the positive root. The velocity there is $20 - 9.8(4.155) \approx -20.72$ m/s, a speed of 20.72 m/s, greater than the launch speed because the ball fell 1.5 m farther than it rose. Energy conservation confirms it: $v^2 = 20^2 + 2(9.8)(1.5) = 429.4$, and $\sqrt{429.4} = 20.72$.

EXERCISES

A. COMPUTATION

3.1 Use the definition to find $f'(a)$ for $f(x) = 3x^2 - 5x + 1$.

3.2 Use the definition to find $f'(x)$ for $f(x) = \frac{1}{x+2}$.

- 3.3** Use the definition to find $f'(x)$ for $f(x) = \sqrt{2x+1}$.
- 3.4** Use the definition to find $f'(x)$ for $f(x) = x^3$.
- 3.5** Find the tangent line to $y = x^3 - 2x$ at $(1, -1)$.
- 3.6** Find the points on $y = x^3 - 3x^2 - 9x + 5$ where the tangent line is horizontal.
- 3.7** Each limit is a derivative $f'(a)$. Identify f and a , and evaluate:
 $\lim_{h \rightarrow 0} \frac{(1+h)^{10} - 1}{h}, \lim_{x \rightarrow \pi/4} \frac{\tan x - 1}{x - \pi/4}.$
- 3.8** Differentiate $y = 5x^4 - 3x^2 + \frac{2}{x} - \sqrt{x}$.
- 3.9** Differentiate $y = \frac{x^3 - 2\sqrt{x}}{x}$.
- 3.10** Differentiate $y = (3x^2 - 1)(x^4 + 2x)$.
- 3.11** Differentiate $y = \frac{2x+3}{x^2-1}$.
- 3.12** Differentiate $y = \frac{\sqrt{x}}{1+\sqrt{x}}$.
- 3.13** Differentiate $y = x^3e^x$ and $y = \frac{e^x}{x^2}$.
- 3.14** Differentiate $y = e^x \sin x$ and $y = \frac{\sin x}{1+\cos x}$.
- 3.15** Differentiate $y = x \tan x - \sec x$.
- 3.16** Find $f''(x)$ for $f(x) = x^4e^x$.
- 3.17** Find $\frac{d^2y}{dx^2}$ for $y = \frac{x}{x+1}$.
- 3.18** Find $\frac{d^n}{dx^n} \left(\frac{1}{x} \right)$ and prove your formula by induction.
- 3.19** Find $\frac{d^{35}}{dx^{35}} \cos x$ and $\frac{d^3}{dx^3} (x^2 \cos x)$.
- 3.20** Find the equations of the tangent and normal lines to $y = \frac{x^2}{x+1}$ at $x = 1$.
- 3.21** Find the points on $y = \frac{1}{x}$ where the tangent line is parallel to $x + 4y = 7$.

- 3.22** For what values of x does $y = x + 2 \sin x$ have a horizontal tangent?
- 3.23** A particle moves with $s(t) = t^3 - 9t^2 + 24t$ for $t \geq 0$. When is it at rest? When does it move backward? Find the total distance traveled in the first 6 seconds.
- 3.24** The height of a rock thrown upward on Mars is $h(t) = 15t - 1.86t^2$ m. Find its velocity after 2 s, its maximum height, and when it lands.
- 3.25** The cost of producing q units is $C(q) = 2000 + 12q - 0.03q^2 + 0.0001q^3$ dollars. Find the marginal cost at $q = 100$ and compare it with $C(101) - C(100)$.
- 3.26** Boyle's law says $PV = c$ at constant temperature. Find the rate of change of V with respect to P , and show that it equals $-\frac{V}{P}$.
- 3.27** Find a, b so that $f(x) = \begin{cases} ax^2 + b, & x \leq 2 \\ x^3, & x > 2 \end{cases}$ is differentiable at 2.
- 3.28** Where is $f(x) = |x - 1| + |x + 1|$ not differentiable? Sketch f' .
- 3.29** Let $f(x) = x|x|$. Find $f'(x)$ for all x . Is f' differentiable at 0?
- 3.30** If $f(2) = 3$, $f'(2) = -1$, $g(2) = 5$, $g'(2) = 4$, find $(fg)'(2)$, $(f/g)'(2)$, and $(f^2 + 3g)'(2)$.

B. THEORY AND PROOF

- 3.31** Prove that the derivative of an even differentiable function is odd, and of an odd one is even.
- 3.32** Prove the power rule for negative integers, $\frac{d}{dx}x^{-n} = -nx^{-n-1}$, using the quotient rule.
- 3.33** Prove that $(fgh)' = f'gh + fg'h + fgh'$, and generalize to n factors.
- 3.34** Prove that if f is differentiable at a , then $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$. Give an example where this limit exists but $f'(a)$ does not.
- 3.35** Show that the tangent line to $xy = 1$ at any point forms, with the coordinate axes, a triangle of area 2, and that the point of tangency is the midpoint of the hypotenuse.
- 3.36** Show that the tangent to $y = x^3$ at $x = a \neq 0$ meets the curve again at $x = -2a$.

- 3.37** Prove that if f is differentiable at a , then $f(x) = f(a) + f'(a)(x - a) + \varepsilon(x)(x - a)$ with $\varepsilon(x) \rightarrow 0$ as $x \rightarrow a$. Conversely, show that if such a representation exists for some number m in place of $f'(a)$, then f is differentiable at a with $f'(a) = m$.
- 3.38** Derive $\frac{d}{dx} \csc x = -\csc x \cot x$ and $\frac{d}{dx} \cot x = -\csc^2 x$.
- 3.39** Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable everywhere, but f' is not continuous at 0.

C. CHALLENGE

- 3.40** Show that $y = x^3 - 3x$ has a tangent line through every point of the plane, and determine how many tangents pass through $(1, a)$ for each a .
- 3.41** Prove that if p is a polynomial and $p(a) = p'(a) = 0$, then $(x - a)^2$ divides $p(x)$. (A double root is a root of the derivative.)
- 3.42** Find all differentiable f with $f(x + y) = f(x)f(y)$ for all x, y and $f'(0) = 1$. (Show $f' = f$.)
- 3.43** The parabola $y = x^2$ reflects any vertical ray to the focus $(0, \frac{1}{4})$. Prove this using the tangent line and the law of reflection.
- 3.44** Using the binomial theorem, show that $(1 + \frac{1}{n})^n$ increases with n and is less than 3 for all n . (This is the first step toward proving that $\lim (1 + \frac{1}{n})^n$ exists; see Chapter 12.)

CHAPTER 4

THE CHAIN RULE AND ITS CONSEQUENCES

LEARNING OBJECTIVES

- State and prove the chain rule, and differentiate compositions of any depth.
- Differentiate implicitly defined functions and find tangent lines to curves given by equations.
- Prove the derivative formula for inverse functions, and differentiate logarithms, inverse trigonometric, and inverse hyperbolic functions.
- Use logarithmic differentiation, and prove the power rule for all real exponents.
- Solve related-rates problems, and use linear approximation and differentials to estimate values and errors.

Most functions met in practice are compositions: $\sin(x^2)$, $e^{-t/\tau}$, $\sqrt{1 - v^2/c^2}$. The chain rule differentiates them, and in doing so unlocks everything else in this chapter: implicit differentiation, the derivatives of inverse functions and logarithms, related rates, and linear approximation. It is the most important rule of differentiation.

4.1 THE CHAIN RULE

Theorem 4.1 (Chain rule). *If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and*

$$(f \circ g)'(x) = f'(g(x)) g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

The Leibniz form looks like cancellation of fractions, and that is the right intuition: if u changes 3 times as fast as x , and y changes 2 times as fast as u , then y changes 6 times as fast as x . The tempting proof, $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$, fails when $\Delta u = 0$ for $\Delta x \neq 0$, which can happen infinitely often near x (as for $g(x) = x^2 \sin \frac{1}{x}$ near 0). The following proof avoids division by Δu .

Proof. Let $b = g(x)$. Define $\varepsilon(k) = \frac{f(b+k) - f(b)}{k} - f'(b)$ for $k \neq 0$ and $\varepsilon(0) = 0$. Then ε is continuous at 0 by differentiability of f , and

$$f(b+k) - f(b) = (f'(b) + \varepsilon(k))k \quad \text{for all } k, \text{ including } k = 0.$$

Put $k = g(x+h) - g(x)$, which tends to 0 as $h \rightarrow 0$ by Theorem 3.4. Then

$$\frac{f(g(x+h)) - f(g(x))}{h} = (f'(b) + \varepsilon(k)) \frac{g(x+h) - g(x)}{h} \rightarrow (f'(b) + 0)g'(x),$$

since $\varepsilon(k) \rightarrow \varepsilon(0) = 0$ by continuity of ε at 0 and of k in h . ■

Example 4.2 (Inside and outside). Differentiate (a) $(x^3 - 2x + 5)^{10}$, (b) $\sin(x^2)$, (c) $e^{-3x} \cos 4x$, (d) $\sqrt{1 + \sqrt{x}}$.

Solution. Differentiate the outer function, evaluated at the inner one, then multiply by the derivative of the inner function.

(a) $10(x^3 - 2x + 5)^9(3x^2 - 2)$. (b) $\cos(x^2) \cdot 2x$. (c) By the product rule and the chain rule twice, $-3e^{-3x} \cos 4x - 4e^{-3x} \sin 4x = -e^{-3x}(3 \cos 4x + 4 \sin 4x)$. (d) $\frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$.

COMMON ERROR. Forgetting the inner derivative: writing $\frac{d}{dx} \sin(x^2) = \cos(x^2)$. Or applying the chain rule where it does not belong: $\frac{d}{dx} (\sin x)^2$ is $2 \sin x \cos x$, while $\frac{d}{dx} \sin(x^2)$ is $2x \cos(x^2)$. Before differentiating, name the outermost operation.

For a composition of three functions the rule is applied twice: $\frac{d}{dx} f(g(h(x))) = f'(g(h(x))) g'(h(x)) h'(x)$. In Leibniz notation, $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$: a chain of rates, which is the origin of the name.

Example 4.3 (A deep composition). Differentiate $y = \sin^3(e^{2x}) = (\sin(e^{2x}))^3$.

Solution. Working from the outside in: $y' = 3 \sin^2(e^{2x}) \cdot \cos(e^{2x}) \cdot e^{2x} \cdot 2 = 6e^{2x} \sin^2(e^{2x}) \cos(e^{2x})$.

Example 4.4 (Exponentials with any base). Since $b^x = e^{x \ln b}$, the chain rule gives $\frac{d}{dx} b^x = e^{x \ln b} \ln b = b^x \ln b$. This confirms the claim of Chapter 3 that the slope of b^x at 0 is $\ln b$. For example, $\frac{d}{dx} 2^x = 2^x \ln 2 \approx 0.693 \cdot 2^x$, and $\frac{d}{dx} 10^{-x} = -10^{-x} \ln 10$.

4.2 IMPLICIT DIFFERENTIATION

Many curves are described by equations rather than formulas: the circle $x^2 + y^2 = 25$, the folium $x^3 + y^3 = 6xy$. Near most of its points such a curve is the graph of a differentiable function $y(x)$, even when no formula for y can be found. (The implicit function theorem of Volume IV makes this precise.) We can still find $\frac{dy}{dx}$: differentiate both sides of the equation with respect to x , treating y as a function of x and using the chain rule on every term containing y , then solve for y' .

Example 4.5 (A circle). Find y' on $x^2 + y^2 = 25$ and the tangent at $(3, -4)$.

Solution. Differentiate: $2x + 2yy' = 0$, so $y' = -\frac{x}{y}$. At $(3, -4)$, $y' = \frac{3}{4}$, and the tangent is $y = -4 + \frac{3}{4}(x - 3)$. Note that $y' = -\frac{x}{y}$ says the tangent is perpendicular to the radius, whose slope is $\frac{y}{x}$: a fact from geometry, recovered without solving for y . At $(\pm 5, 0)$ the formula fails, and the tangents are vertical.

Example 4.6 (The folium of Descartes). Find y' on $x^3 + y^3 = 6xy$, the tangent at $(3, 3)$, and the point in the first quadrant where the tangent is horizontal.

Solution. Differentiate: $3x^2 + 3y^2y' = 6y + 6xy'$, so $y'(y^2 - 2x) = 2y - x^2$ and

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

At $(3, 3)$: $y' = \frac{6-9}{9-6} = -1$, and the tangent is $x + y = 6$. The tangent is horizontal where $2y = x^2$, with (x, y) on the curve: $x^3 + \frac{x^6}{8} = 6x \cdot \frac{x^2}{2}$, so $x^6 = 16x^3$ and (for $x \neq 0$) $x^3 = 16$, $x = 2^{4/3} \approx 2.520$, $y = \frac{x^2}{2} = 2^{5/3} \approx 3.175$.

Example 4.7 (Second derivative implicitly). Find y'' on $x^4 + y^4 = 16$.

Solution. First $4x^3 + 4y^3y' = 0$, so $y' = -\frac{x^3}{y^3}$. Differentiate this with the quotient rule, remembering that y depends on x :

$$\begin{aligned} y'' &= -\frac{3x^2y^3 - x^3 \cdot 3y^2y'}{y^6} = -\frac{3x^2y - 3x^3y'}{y^4} \\ &= -\frac{3x^2y + 3x^6/y^3}{y^4} = -\frac{3x^2(y^4 + x^4)}{y^7} = -\frac{48x^2}{y^7}, \end{aligned}$$

using the original equation $x^4 + y^4 = 16$ at the end. Simplifying with the equation of the curve is often the key step.

4.2.1 THE POWER RULE FOR RATIONAL EXPONENTS

If $y = x^{p/q}$, then $y^q = x^p$. Differentiating implicitly, $qy^{q-1}y' = px^{p-1}$, so

$$y' = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p(q-1)/q}} = \frac{p}{q} x^{p/q-1}.$$

This assumes y is differentiable, which follows from the theorem on inverse functions below.

4.3 DERIVATIVES OF INVERSE FUNCTIONS AND LOGARITHMS

Theorem 4.8 (Derivative of an inverse function). Let f be continuous and one-to-one on an open interval, differentiable at a with $f'(a) \neq 0$, and let $b = f(a)$. Then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Proof. A continuous one-to-one function on an interval is monotone, and its in-

verse is continuous (Appendix A). Let $k \neq 0$ be small and $h = f^{-1}(b+k) - a$, so $f(a+h) = b+k$. Then $h \neq 0$ (since f^{-1} is one-to-one), and $h \rightarrow 0$ as $k \rightarrow 0$ (since f^{-1} is continuous). Hence

$$\frac{f^{-1}(b+k) - f^{-1}(b)}{k} = \frac{h}{f(a+h) - f(a)} = \frac{1}{\frac{f(a+h) - f(a)}{h}} \rightarrow \frac{1}{f'(a)}.$$

Geometrically: the graph of f^{-1} is the reflection of the graph of f in $y = x$, and reflection inverts slopes. In Leibniz notation, $\frac{dx}{dy} = \frac{1}{dy/dx}$. Where $f'(a) = 0$, the reflected tangent is vertical, and f^{-1} is not differentiable: x^3 has slope 0 at 0, and its inverse $\sqrt[3]{x}$ has a vertical tangent there.

Example 4.9 (Without a formula for the inverse). Let $f(x) = x^5 + x + 1$. Find $(f^{-1})'(3)$.

Solution. $f'(x) = 5x^4 + 1 > 0$, so f is increasing and one-to-one. We need a with $f(a) = 3$: by inspection $a = 1$. Then $(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{6}$. No formula for f^{-1} exists in terms of radicals, but its derivative at this point is exact.

Theorem 4.10 (Derivative of the logarithm). $\frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$, and $\frac{d}{dx} \ln |x| = \frac{1}{x}$ for $x \neq 0$. More generally $\frac{d}{dx} \log_b x = \frac{1}{x \ln b}$.

Proof. $y = \ln x$ means $e^y = x$. By Theorem 4.8, $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}$. For $x < 0$, $\ln |x| = \ln(-x)$, whose derivative by the chain rule is $\frac{1}{-x} \cdot (-1) = \frac{1}{x}$. Finally $\log_b x = \frac{\ln x}{\ln b}$.

Example 4.11 (Logarithms with the chain rule). Differentiate $\ln(x^2 + 1)$, $\ln(\cos x)$, and $\ln \sqrt{\frac{1+x}{1-x}}$.

Solution. $\frac{2x}{x^2+1} \cdot \frac{-\sin x}{\cos x} = -\tan x$. For the third, simplify first with the laws of logarithms: $\frac{1}{2}(\ln(1+x) - \ln(1-x))$ has derivative $\frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{1-x^2}$. (This is $\tanh^{-1} x$, by the last worked problem of Chapter 1.)

4.3.1 LOGARITHMIC DIFFERENTIATION

To differentiate a complicated product, quotient, or power, take logarithms first.

If $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$, then

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2), \quad \frac{y'}{y} = \frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2},$$

and y' is y times the right side. The quantity $\frac{y'}{y} = (\ln y)'$ is the *logarithmic derivative*, the relative rate of change of y .

Theorem 4.12 (General power rule). For any real r , $\frac{d}{dx} x^r = r x^{r-1}$ for $x > 0$.

Proof. $x^r = e^{r \ln x}$, so its derivative is $e^{r \ln x} \cdot \frac{r}{x} = x^r \cdot \frac{r}{x} = r x^{r-1}$. ■

Example 4.13 (Variable base and exponent). Differentiate $y = x^x$ for $x > 0$.

Solution. Neither the power rule (variable base) nor the exponential rule (variable exponent) applies. Take logarithms: $\ln y = x \ln x$, so $\frac{y'}{y} = \ln x + 1$ and $y' = x^x(1 + \ln x)$. The minimum of x^x is at $x = \frac{1}{e}$, where $y' = 0$.

Theorem 4.14 (The limit defining e). $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$, and in particular

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Proof. $f(x) = \ln x$ has $f'(1) = 1$, so $\lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln 1}{x} = 1$, that is, $\ln((1+x)^{1/x}) \rightarrow 1$. Since e^u is continuous, $(1+x)^{1/x} = e^{\ln((1+x)^{1/x})} \rightarrow e^1$. Taking $x = \frac{1}{n}$ gives the second limit. ■

The same argument gives $\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r$: interest at annual rate r , compounded n times a year, grows by the factor $\left(1 + \frac{r}{n}\right)^n$, which tends to e^r under continuous compounding.

4.4 INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Theorem 4.15.

$$\begin{aligned}\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}} \quad (|x| < 1), \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2}, & \frac{d}{dx} \sinh x &= \cosh x, & \frac{d}{dx} \cosh x &= \sinh x, \\ \frac{d}{dx} \tanh x &= \operatorname{sech}^2 x, & \frac{d}{dx} \sinh^{-1} x &= \frac{1}{\sqrt{1+x^2}}, \\ \frac{d}{dx} \cosh^{-1} x &= \frac{1}{\sqrt{x^2-1}} \quad (x > 1), & \frac{d}{dx} \tanh^{-1} x &= \frac{1}{1-x^2} \quad (|x| < 1).\end{aligned}$$

Proof. For $y = \arcsin x$: $\sin y = x$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, where $\cos y \geq 0$. By Theorem 4.8, $y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$; the restricted range is what selects the positive root. For $y = \arctan x$: $y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$. Since $\arccos x = \frac{\pi}{2} - \arcsin x$ (Exercise 1.39), its derivative is the negative of that of $\arcsin$. The hyperbolic derivatives follow from $\frac{d}{dx} e^{\pm x} = \pm e^{\pm x}$, and their inverses from the logarithmic formulas of Chapter 1 or from Theorem 4.8 with $\cosh^2 - \sinh^2 = 1$. ■

The derivatives of the transcendental functions $\arctan$ and $\arcsin$ are algebraic. Read backward, this means the integrals of $\frac{1}{1+x^2}$ and $\frac{1}{\sqrt{1-x^2}}$, which are algebraic functions, are transcendental. That is the origin of π in so many integrals.

Example 4.16 (Angles of elevation). Differentiate $\theta(x) = \arctan \frac{x}{2} - \arctan \frac{x}{5}$, and interpret.

Solution. $\theta' = \frac{1/2}{1+x^2/4} - \frac{1/5}{1+x^2/25} = \frac{2}{4+x^2} - \frac{5}{25+x^2} = \frac{2(25+x^2)-5(4+x^2)}{(4+x^2)(25+x^2)} = \frac{30-3x^2}{(4+x^2)(25+x^2)}$, zero at $x = \sqrt{10}$. If a viewer at horizontal distance x from a wall looks at a painting whose bottom is 2 m and top 5 m above eye level, then θ is the angle the painting subtends, and $\sqrt{10} \approx 3.16$ m is the distance at which it appears largest: the geometric mean of 2 and 5.

4.5 RELATED RATES

When several quantities are linked by an equation and all change with time, differentiating the equation with respect to t links their rates of change. A systematic procedure:

1. Draw a diagram and name every quantity that varies with time. Do not substitute numerical values for them yet.
2. Write down the given rates and the rate to be found, as derivatives with respect to t .
3. Find an equation relating the quantities, valid at all times.
4. Differentiate it with respect to t , using the chain rule.
5. Only now substitute the values at the instant of interest, and solve for the unknown rate.

Example 4.17 (A sliding ladder). A 10 m ladder leans against a wall. Its foot slides away from the wall at 1 m/s. How fast is the top sliding down when the foot is 6 m from the wall? How fast is the angle between the ladder and the ground changing?

Solution. Let x be the distance from the wall to the foot and y the height of the top. Then $x^2 + y^2 = 100$ at all times, and $\frac{dx}{dt} = 1$. Differentiating, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $x = 6$, $y = 8$, so $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{8} = -0.75$ m/s: the top slides down at 0.75 m/s.

For the angle θ , $\cos \theta = \frac{x}{10}$. Differentiating, $-\sin \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt}$. At the instant, $\sin \theta = \frac{8}{10}$, so $\frac{d\theta}{dt} = -\frac{1}{8}$ rad/s.

Note that $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$ grows without bound as $y \rightarrow 0$: the model predicts the top falling infinitely fast at the end. In reality the top leaves the wall before that, and the equation $x^2 + y^2 = 100$ stops describing the motion.

Example 4.18 (Filling a cone). Water flows at $2 \text{ m}^3/\text{min}$ into a tank shaped as an inverted cone, 4 m deep with top radius 2 m. How fast is the level rising when the water is 3 m deep?

Solution. At depth h the water surface has radius r , and similar triangles give $\frac{r}{h} = \frac{2}{4}$, so $r = \frac{h}{2}$. The volume is $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{12}h^3$. Differentiating, $\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$. With $\frac{dV}{dt} = 2$ and $h = 3$: $\frac{dh}{dt} = \frac{8}{9\pi} \approx 0.283$ m/min. Eliminating r before differentiating avoided needing $\frac{dr}{dt}$.

Example 4.19 (Two cars). Car A drives west at 90 km/h and car B north at 100 km/h, both toward the intersection of their roads. How fast are they approaching each other when A is 0.3 km and B 0.4 km from the intersection?

Solution. With x, y the distances to the intersection and z the distance between the cars, $z^2 = x^2 + y^2$, and $2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$. Both distances are decreasing: $\frac{dx}{dt} = -90$, $\frac{dy}{dt} = -100$. At the instant $z = 0.5$, so $\frac{dz}{dt} = \frac{0.3(-90) + 0.4(-100)}{0.5} = -134$ km/h. They approach at 134 km/h.

4.6 LINEAR APPROXIMATION AND DIFFERENTIALS

Near a point where f is differentiable, its graph is nearly indistinguishable from its tangent line. The *linearization* of f at a is

$$L(x) = f(a) + f'(a)(x - a),$$

and the *linear approximation* is $f(x) \approx L(x)$ for x near a . By the definition of the derivative, the error $f(x) - L(x)$ is small compared with $x - a$: $\frac{f(x) - L(x)}{x - a} \rightarrow 0$. When f'' exists, the error is of order $(x - a)^2$; Taylor's theorem in Chapter 13 gives the precise bound $|f(x) - L(x)| \leq \frac{M}{2}(x - a)^2$, where M bounds $|f''|$ between a and x .

Example 4.20 (Square roots and sines). Approximate $\sqrt{4.1}$, $\sqrt[3]{26.5}$, and $\sin(31^\circ)$.

Solution. $\sqrt{x}$ at $a = 4$: $L(x) = 2 + \frac{1}{4}(x - 4)$, so $\sqrt{4.1} \approx 2.025$. (True value 2.024846.)

$\sqrt[3]{x}$ at $a = 27$: $f'(27) = \frac{1}{3} \cdot 27^{-2/3} = \frac{1}{27}$, so $\sqrt[3]{26.5} \approx 3 - \frac{0.5}{27} = 2.98148$. (True value 2.98137.)

$\sin x$ at $a = \frac{\pi}{6}$: $31^\circ = \frac{\pi}{6} + \frac{\pi}{180}$, so $\sin 31^\circ \approx \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180} = 0.51511$. (True value 0.51504.) The increment must be in radians.

Example 4.21 (Approximations behind physics). Near 0: $\sin x \approx x$, $\cos x \approx 1$, $\tan x \approx x$, $e^x \approx 1 + x$, $\ln(1 + x) \approx x$, and $(1 + x)^r \approx 1 + rx$. The last, with $r = -\frac{1}{2}$, gives the low-speed limit of relativity: $\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{v^2}{2c^2}$, so the relativistic energy $\frac{mc^2}{\sqrt{1 - v^2/c^2}} \approx mc^2 + \frac{1}{2}mv^2$, rest energy plus the classical kinetic energy. The approximation $\sin \theta \approx \theta$ turns the pendulum equation into that of simple harmonic motion.

4.6.1 DIFFERENTIALS

If $y = f(x)$, the *differential* dy is defined as $dy = f'(x) dx$, where dx is an independent variable (any real number). Geometrically, dx is a change in x , the actual change in y is $\Delta y = f(x + dx) - f(x)$, and dy is the change along the tangent line. For small dx , $\Delta y \approx dy$. Differentials make the Leibniz notation $\frac{dy}{dx}$ an honest quotient, and they are the natural language for error estimates.

Example 4.22 (Propagation of error). The radius of a ball is measured as 21 cm with a possible error of 0.05 cm. Estimate the maximum error in the computed volume, and the relative error.

Solution. $V = \frac{4}{3}\pi r^3$, so $dV = 4\pi r^2 dr$. With $r = 21$ and $|dr| \leq 0.05$: $|dV| \leq 4\pi(441)(0.05) \approx 277 \text{ cm}^3$. The volume is about 38792 cm^3 , so the relative error is $\frac{dV}{V} = 3\frac{dr}{r} \approx 3(0.00238) = 0.714\%$. The relative error in a power r^n is n times the relative error in r : measuring a radius to 1% gives a volume to only 3%.

4.7 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— **Problem 4.1** A logarithmic derivative

Differentiate $y = \frac{(x^2 + 1)^3 \sqrt{2x - 1}}{(x + 4)^5}$, and find where $y' = 0$ for $x > \frac{1}{2}$.

Solution. Take logarithms: $\ln y = 3 \ln(x^2 + 1) + \frac{1}{2} \ln(2x - 1) - 5 \ln(x + 4)$. Differentiating,

$$\frac{y'}{y} = \frac{6x}{x^2 + 1} + \frac{1}{2x - 1} - \frac{5}{x + 4}.$$

Since $y \neq 0$ for $x > \frac{1}{2}$, $y' = 0$ exactly when the right side vanishes. Over the common denominator $(x^2 + 1)(2x - 1)(x + 4)$, the numerator is

$$6x(2x - 1)(x + 4) + (x^2 + 1)(x + 4) - 5(x^2 + 1)(2x - 1) = 3x^3 + 51x^2 - 33x + 9.$$

A cubic with positive leading coefficient, it has one real root, $x \approx -17.6$, outside the domain; on $x > \frac{1}{2}$ the numerator is positive (it increases there, from 5.625 at $x = \frac{1}{2}$). So $y' > 0$ for all $x > \frac{1}{2}$, and y is increasing. Logarithmic differentiation turned a quotient-rule nightmare into a sum of simple fractions.

— **Problem 4.2** Implicit differentiation and a tangent line

For the curve $x^2y + y^3 = 10$, find y' , the tangent line at $(1, 2)$, and y'' at that point.

Solution. Differentiate: $2xy + x^2y' + 3y^2y' = 0$, so $y' = -\frac{2xy}{x^2 + 3y^2}$. At $(1, 2)$, $y' = -\frac{4}{13}$, and the tangent is $y = 2 - \frac{4}{13}(x - 1)$. For y'' , differentiate $2xy + (x^2 + 3y^2)y' = 0$ again:

$$2y + 2xy' + (2x + 6yy')y' + (x^2 + 3y^2)y'' = 0.$$

At $(1, 2)$ with $y' = -\frac{4}{13}$: $4 - \frac{8}{13} + (2 - \frac{48}{13})(-\frac{4}{13}) + 13y'' = 0$. The middle terms give $4 - \frac{8}{13} + \frac{88}{169} = \frac{676 - 104 + 88}{169} = \frac{660}{169}$, so $y'' = -\frac{660}{2197} \approx -0.300$. Differentiating the unsolved equation, rather than the formula for y' , avoided a quotient rule.

— **Problem 4.3** An inverse function's second derivative

If f is twice differentiable and one-to-one with $f' \neq 0$, and $g = f^{-1}$, show that $g''(y) = -\frac{f''(x)}{f'(x)^3}$ where $x = g(y)$. Apply it to $f(x) = x + e^x$ at $y = 1$.

Solution. $g'(y) = \frac{1}{f'(g(y))}$. Differentiate with the chain rule: $g''(y) = -\frac{f''(g(y))g'(y)}{f'(g(y))^2} = -\frac{f''(x)}{f'(x)^3}$, using $g'(y) = \frac{1}{f'(x)}$.

For $f(x) = x + e^x$: $f(0) = 1$, so $x = 0$ when $y = 1$. Then $f'(0) = 2$ and $f''(0) = 1$, giving $g'(1) = \frac{1}{2}$ and $g''(1) = -\frac{1}{8}$. The inverse of $x + e^x$ (expressible only through the Lambert W function) has a known slope and curvature at 1 without any formula.

— **Problem 4.4** A shadow

A man 1.8 m tall walks at 1.5 m/s away from a street light 6 m high. How fast is his shadow lengthening, and how fast is the tip of the shadow moving?

Solution. Let x be his distance from the base of the light and s the length of his shadow. Similar triangles (light, ground, tip; man, ground, tip) give $\frac{6}{x+s} = \frac{1.8}{s}$, so $6s = 1.8x + 1.8s$ and $s = \frac{1.8}{4.2}x = \frac{3}{7}x$. Then $\frac{ds}{dt} = \frac{3}{7} \cdot 1.5 = \frac{9}{14} \approx 0.643$ m/s. The tip is at $x + s = \frac{10}{7}x$, moving at $\frac{10}{7} \cdot 1.5 = \frac{15}{7} \approx 2.14$ m/s. Both rates are constant, independent of his position: the relations are linear, so the derivatives are constant multiples of $\frac{dx}{dt}$.

— **Problem 4.5** The angle of a rising balloon

A balloon rises vertically at 3 m/s from a point 40 m from an observer. How fast

is the angle of elevation changing when the balloon is 30 m high? When is that rate greatest?

Solution. With height h and angle θ , $\tan \theta = \frac{h}{40}$. Differentiate: $\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{40} \frac{dh}{dt}$. At $h = 30$, the line of sight has length 50, so $\sec \theta = \frac{5}{4}$ and $\frac{d\theta}{dt} = \frac{3}{40} \cdot \frac{16}{25} = 0.048$ rad/s. In general $\theta = \arctan \frac{h}{40}$, so $\frac{d\theta}{dt} = \frac{40}{1600+h^2} \frac{dh}{dt}$, which is greatest at $h = 0$, where it is $\frac{3}{40} = 0.075$ rad/s. The angle changes fastest at the start and ever more slowly as the balloon rises, approaching $\frac{\pi}{2}$.

— Problem 4.6 Linear approximation with an error bound

Use the linearization of $\sqrt{x}$ at 100 to approximate $\sqrt{98}$, and bound the error using the fact that $|f(x) - L(x)| \leq \frac{M}{2}(x - a)^2$ when $|f''| \leq M$ between a and x .

Solution. $L(x) = 10 + \frac{1}{20}(x - 100)$, so $\sqrt{98} \approx 10 - 0.1 = 9.9$. For the error, $f''(x) = -\frac{1}{4}x^{-3/2}$, whose absolute value on $[98, 100]$ is largest at 98: $M = \frac{1}{4}(98)^{-3/2} < \frac{1}{4}(97)^{-3/2} \approx 2.6 \times 10^{-4}$. So the error is at most $\frac{2.6 \times 10^{-4}}{2} \cdot 4 \approx 5.2 \times 10^{-4}$. The true value is 9.89949, an error of 5.1×10^{-4} , inside the bound. Since $f'' < 0$, the graph is concave down and the tangent line lies above it, so the approximation is an overestimate, as observed.

— Problem 4.7 Differentiating x^{x^x}

Find the derivative of $y = x^{x^x} = x^{(x^x)}$ for $x > 0$.

Solution. Take logarithms: $\ln y = x^x \ln x$. Differentiate the right side by the product rule, using $(x^x)' = x^x(1 + \ln x)$ from Example 4.13:

$$\frac{y'}{y} = x^x(1 + \ln x) \ln x + x^x \cdot \frac{1}{x} = x^x \left(\ln x + \ln^2 x + \frac{1}{x} \right).$$

So $y' = x^{x^x} x^x \left(\ln x + \ln^2 x + \frac{1}{x} \right)$. At $x = 1$, $y' = 1 \cdot 1 \cdot (0 + 0 + 1) = 1$. Exponent towers are evaluated from the top: x^{x^x} means $x^{(x^x)}$, not $(x^x)^x = x^{x^2}$, whose derivative is $x^{x^2}(2x \ln x + x)$.

EXERCISES

A. COMPUTATION

4.1 Differentiate $y = (2x^3 - 5)^7$ and $y = \frac{1}{(x^2 + 4)^3}$.

4.2 Differentiate $y = \sqrt{x^2 + 3x}$ and $y = \sqrt[3]{1 - x^4}$.

- 4.3** Differentiate $y = \sin(3x^2 + 1)$ and $y = \cos^4(2x)$.
- 4.4** Differentiate $y = \tan(e^x)$ and $y = e^{\tan x}$.
- 4.5** Differentiate $y = x^2 e^{-x^2}$.
- 4.6** Differentiate $y = \left(\frac{x-1}{x+1}\right)^4$.
- 4.7** Differentiate $y = \sec^2(5x) - \tan^2(5x)$, and explain the result.
- 4.8** Differentiate $y = \ln(x^3 - 2x)$ and $y = \ln(\sec x + \tan x)$.
- 4.9** Differentiate $y = \log_2(x^2 + 1)$ and $y = 3^{\sin x}$.
- 4.10** Differentiate $y = \arcsin(2x)$ and $y = \arctan \sqrt{x}$.
- 4.11** Differentiate $y = x \arctan x - \frac{1}{2} \ln(1 + x^2)$.
- 4.12** Differentiate $y = \sinh(3x) \cosh(2x)$ and $y = \tanh^{-1}(\sin x)$.
- 4.13** Use logarithmic differentiation: $y = \frac{x^2 \sqrt{x+1}}{(x-2)^3}$.
- 4.14** Differentiate $y = x^{\sin x}$ and $y = (\ln x)^x$.
- 4.15** Find y' if $x^2 + xy + y^2 = 7$, and the tangent at $(1, 2)$.
- 4.16** Find y' if $\sin(xy) = x$.
- 4.17** Find y' if $e^{x+y} = xy$.
- 4.18** Find y'' if $x^2 - y^2 = 1$.
- 4.19** Find the points on $x^2 - xy + y^2 = 3$ where the tangent is horizontal, and where it is vertical.
- 4.20** If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.
- 4.21** Find $(f^{-1})'(2)$ for $f(x) = x^3 + 3x$.
- 4.22** A spherical balloon is inflated at $50 \text{ cm}^3/\text{s}$. How fast is the radius increasing when the radius is 10 cm ?
- 4.23** A 5 m ladder slides down a wall. When the foot is 3 m out, it moves away at 0.4 m/s . How fast does the top move?
- 4.24** A trough 6 m long has triangular cross-sections 1 m across the top and 0.5 m deep. Water enters at $0.2 \text{ m}^3/\text{min}$. How fast does the level rise when the water is 0.3 m deep?

- 4.25** A plane flying horizontally at 9 km altitude at 600 km/h passes directly over a radar station. How fast is its distance from the station increasing when that distance is 15 km?
- 4.26** A point moves on $y = \sqrt{x}$ so that x increases at 3 units/s. How fast is its distance from the origin changing at $x = 4$?
- 4.27** Find the linearization of $f(x) = \ln x$ at $x = 1$, and use it to approximate $\ln 1.05$.
- 4.28** Approximate $\sqrt{37}$ and $(1.98)^5$ by linear approximation.
- 4.29** The side of a cube is measured as 30 cm with possible error 0.1 cm. Estimate the maximum error and relative error in its volume and in its surface area.
- 4.30** The period of a pendulum is $T = 2\pi\sqrt{L/g}$. If L increases by 2%, by about what percentage does T change?

B. THEORY AND PROOF

- 4.31** Prove that if f is differentiable and even, then f' is odd, using the chain rule on $f(-x) = f(x)$.
- 4.32** Prove that if f is periodic with period p and differentiable, then f' is periodic with period p .
- 4.33** Derive the quotient rule from the product rule and the chain rule applied to $f \cdot (g)^{-1}$.
- 4.34** Derive $\frac{d}{dx} \arccos x$ directly from Theorem 4.8.
- 4.35** Prove $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$ in two ways: from the logarithmic formula, and from Theorem 4.8.
- 4.36** Show that $\frac{d}{dx} \ln |\sec x + \tan x| = \sec x$.
- 4.37** Show that the curves $x^2 - y^2 = a$ and $xy = b$ intersect at right angles for any nonzero a, b (they are *orthogonal trajectories*).
- 4.38** Show that the tangent lines to $\sqrt{x} + \sqrt{y} = \sqrt{c}$ have intercepts summing to c .
- 4.39** Prove that $\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r$.

C. CHALLENGE

- 4.40** Let $f(x) = x^2 \sin \frac{1}{x}$ ($f(0) = 0$) and $g(x) = x$. Verify the chain rule for $f \circ g$ at 0, and explain why the naive proof fails for $g(x) = x^2 \sin \frac{1}{x}$, $f(x) = x$ composed the other way.
- 4.41** The *astroid* $x^{2/3} + y^{2/3} = 1$: show that the segment of every tangent line between the coordinate axes has length 1.
- 4.42** Find all $x > 0$ for which the tangent line to $y = x^x$ passes through the origin.
- 4.43** A lighthouse 1 km offshore has a beam rotating at 2 revolutions per minute. How fast does the beam move along the straight shore at a point 2 km from the point nearest the lighthouse?
- 4.44** Prove that if $y = f(x)$ satisfies $y' = ky$ for a constant k , then $y = Ce^{kx}$. (Differentiate ye^{-kx} .)

CHAPTER 5

THE MEAN VALUE THEOREM AND THE SHAPE OF GRAPHS

LEARNING OBJECTIVES

- Distinguish absolute and local extrema, prove Fermat's theorem, and find absolute extrema on closed intervals.
- State and prove Rolle's theorem and the Mean Value Theorem, and use them to prove inequalities and count roots.
- Use the first derivative to find intervals of increase and decrease and classify critical points.
- Use the second derivative to find concavity and inflection points, and apply the second derivative test.
- Sketch the graph of a function from the information its derivatives provide.

The derivative at a point is local information: a slope at one place. This chapter turns it into global information about the shape of a graph on a whole interval. The bridge is the Mean Value Theorem, which guarantees that an average rate of change is achieved as an instantaneous rate somewhere. From it follow the tests for increase and decrease, for maxima and minima, and for concavity: the tools that let us sketch a graph, prove inequalities, and, in Chapter 6, solve optimization problems.

5.1 MAXIMUM AND MINIMUM VALUES

Definition 5.1 (Extrema). Let c be in the domain D of f . Then $f(c)$ is the *absolute maximum* of f if $f(c) \geq f(x)$ for all $x \in D$, and a *local maximum* if $f(c) \geq f(x)$ for all x in some open interval around c . Minima are defined with the inequalities reversed. Maxima and minima are *extreme values* or *extrema*.

An absolute extremum at an interior point is also local. At an endpoint of a closed interval, a function can have an absolute extremum that is not local in this sense, since no open interval around the endpoint lies in the domain.

Theorem 5.2 (Fermat's theorem). If f has a local maximum or minimum at c , and $f'(c)$ exists, then $f'(c) = 0$.

Proof. Suppose a local maximum. For small $h > 0$, $f(c+h) \leq f(c)$, so $\frac{f(c+h)-f(c)}{h} \leq 0$, and the right-hand limit gives $f'(c) \leq 0$. For small $h < 0$, the same numerator is ≤ 0 but the denominator is negative, so the quotient is ≥ 0 , and the left-hand limit gives $f'(c) \geq 0$. Hence $f'(c) = 0$. For a minimum, apply this to $-f$. ■

Definition 5.3 (Critical number). A *critical number* of f is a number c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.

By Fermat's theorem, every local extremum occurs at a critical number. The converse fails: $f(x) = x^3$ has $f'(0) = 0$ but no extremum at 0. Critical numbers are candidates, not conclusions.

Theorem 5.4 (Closed interval method). If f is continuous on $[a, b]$, its absolute maximum and minimum exist (Theorem 2.30), and they occur at critical numbers in (a, b) or at the endpoints. To find them, evaluate f at all of these points and compare.

Example 5.5 (Extrema on a closed interval). Find the absolute extrema of $f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$, and of $g(x) = x^{2/3}(x - 5)$ on $[-1, 4]$.

Solution. $f'(x) = 3x^2 - 6x = 3x(x - 2)$, zero at 0 and 2. Values: $f(-\frac{1}{2}) = \frac{1}{8}$, $f(0) = 1$, $f(2) = -3$, $f(4) = 17$. The maximum is 17 at $x = 4$ (an

endpoint) and the minimum -3 at $x = 2$.

$g(x) = x^{5/3} - 5x^{2/3}$, so $g'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5(x-2)}{3x^{1/3}}$. Critical numbers: $x = 2$ ($g' = 0$) and $x = 0$ (g' undefined, a cusp). Values: $g(-1) = -6$, $g(0) = 0$, $g(2) = -3 \cdot 2^{2/3} \approx -4.762$, $g(4) = -4^{2/3} \approx -2.520$. The maximum is 0 at the cusp, and the minimum -6 at the endpoint -1 . Omitting the critical number where g' fails to exist would have missed the maximum.

5.2 THE MEAN VALUE THEOREM

Theorem 5.6 (Rolle's theorem). *If f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then there is $c \in (a, b)$ with $f'(c) = 0$.*

Proof. By the Extreme Value Theorem, f attains a maximum and a minimum on $[a, b]$. If both occur at the endpoints, then since $f(a) = f(b)$, the maximum equals the minimum and f is constant, so $f'(c) = 0$ for every c . Otherwise one of them occurs at an interior point c , and by Fermat's theorem $f'(c) = 0$. ■

Theorem 5.7 (Mean Value Theorem). *If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is $c \in (a, b)$ with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Subtract from f the secant line through its endpoints: let $h(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$. Then h is continuous on $[a, b]$, differentiable on (a, b) , and $h(a) = h(b) = 0$. By Rolle's theorem there is c with $0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$. ■

Geometrically, some tangent line is parallel to the secant line. Physically, if you drive 120 km in 1.5 hours, at some instant your speedometer read exactly 80 km/h. The theorem says nothing about where c is; its power lies in its existence. All three hypotheses matter: $|x|$ on $[-1, 1]$ is continuous with equal endpoint values, but no tangent is horizontal, because it is not differentiable at 0.

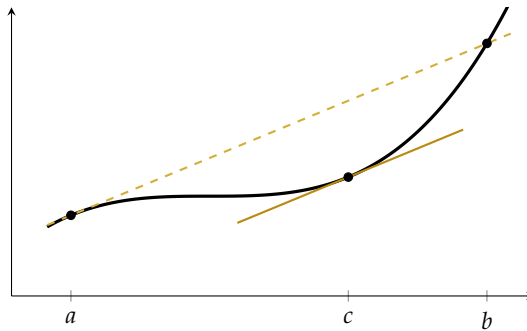


Figure 5.1: The Mean Value Theorem: at some c between a and b , the tangent (solid) is parallel to the secant (dashed).

5.2.1 CONSEQUENCES

Theorem 5.8. If $f'(x) = 0$ for all x in an interval I , then f is constant on I . If $f'(x) = g'(x)$ for all $x \in I$, then $f - g$ is constant on I .

Proof. For any $x_1 < x_2$ in I , the MVT gives c with $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$. Apply this to $f - g$ for the second statement. ■

This innocent theorem is the foundation of integration: two antiderivatives of the same function on an interval differ by a constant. The interval hypothesis matters: $f(x) = \frac{|x|}{x}$ has $f' = 0$ on its domain $\mathbb{R} \setminus \{0\}$, but it is not constant.

Example 5.9 (An identity from a derivative). Prove that $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$.

Solution. Let $f(x) = \arctan x + \arctan \frac{1}{x}$. Then $f'(x) = \frac{1}{1+x^2} + \frac{1}{1+x^{-2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$. On the interval $(0, \infty)$, f is constant, and $f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Compare the trigonometric proof in the worked problems of Chapter 1. On $(-\infty, 0)$ the constant is $-\frac{\pi}{2}$, which shows again why the interval matters.

Example 5.10 (Inequalities from the MVT). Prove that (a) $|\sin a - \sin b| \leq |a - b|$ for all a, b ; (b) $\frac{x}{1+x} < \ln(1+x) < x$ for $x > 0$.

Solution. (a) By the MVT, $\sin a - \sin b = \cos c(a - b)$ for some c , and $|\cos c| \leq 1$. Any function with $|f'| \leq M$ satisfies $|f(a) - f(b)| \leq M|a - b|$:

it is *Lipschitz*.

(b) Apply the MVT to $\ln t$ on $[1, 1+x]$: $\ln(1+x) - \ln 1 = \frac{1}{c} \cdot x$ for some $c \in (1, 1+x)$. Since $\frac{1}{1+x} < \frac{1}{c} < 1$, multiplying by $x > 0$ gives the result. With $x = \frac{1}{n}$: $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$, which will bound the harmonic series in Chapter 12.

Example 5.11 (Counting roots). Show that $x^5 + x^3 + 2x - 1 = 0$ has exactly one real root.

Solution. *At least one:* $f(0) = -1 < 0 < 3 = f(1)$, so the IVT gives a root in $(0, 1)$. *At most one:* if f had two roots $a < b$, Rolle's theorem would give c with $f'(c) = 0$. But $f'(x) = 5x^4 + 3x^2 + 2 \geq 2 > 0$. So there is exactly one root. In general, between two roots of f lies a root of f' , so f has at most one more root than f' .

Theorem 5.12 (Cauchy's Mean Value Theorem). *If f and g are continuous on $[a, b]$ and differentiable on (a, b) , there is $c \in (a, b)$ with $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.*

Proof. Apply Rolle's theorem to $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$, which has $h(a) = f(b)g(a) - g(b)f(a) = h(b)$. ■

With $g(x) = x$ this is the ordinary MVT. For a curve traced by $(g(t), f(t))$, it says some tangent is parallel to the chord. It is the key to L'Hôpital's rule in Chapter 6.

5.3 MONOTONICITY AND THE FIRST DERIVATIVE TEST

Theorem 5.13 (Increasing/decreasing test). *If $f' > 0$ on an interval, f is increasing there; if $f' < 0$, f is decreasing.*

Proof. For $x_1 < x_2$ in the interval, the MVT gives $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, which has the sign of $f'(c)$. ■

The converse is false in its strict form: x^3 is increasing but $f'(0) = 0$. An increasing differentiable function has $f' \geq 0$.

Theorem 5.14 (First derivative test). Let c be a critical number of a function f continuous at c . If f' changes from positive to negative at c , then f has a local maximum at c ; if from negative to positive, a local minimum; if f' does not change sign, no local extremum.

Example 5.15 (Intervals of increase and local extrema). Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increases and decreases, and its local extrema.

Solution. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$, with critical numbers $-1, 0, 2$. A sign chart of the three factors:

Interval	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$x > 2$
Sign of f'	–	+	–	+
f	decreasing	increasing	decreasing	increasing

So $f(-1) = 0$ is a local minimum, $f(0) = 5$ a local maximum, and $f(2) = -27$ a local minimum, which is also the absolute minimum.

5.4 CONCAVITY AND THE SECOND DERIVATIVE

Definition 5.16 (Concavity). f is *concave upward* on an interval if its graph lies above all its tangent lines there, and *concave downward* if below. A point where the concavity changes is an *inflection point*.

Theorem 5.17 (Concavity test). If $f'' > 0$ on an interval, f is concave upward there; if $f'' < 0$, concave downward.

Proof. Suppose $f'' > 0$, so f' is increasing. Fix a and let $L(x) = f(a) + f'(a)(x - a)$. For $x > a$, the MVT gives $c \in (a, x)$ with $f(x) - f(a) = f'(c)(x - a) > f'(a)(x - a)$, since $f'(c) > f'(a)$. So $f(x) > L(x)$. For $x < a$, $c \in (x, a)$ gives $f'(c) < f'(a)$, and multiplying by $x - a < 0$ reverses the inequality to the same conclusion. ■

Theorem 5.18 (Second derivative test). Suppose f'' is continuous near c and $f'(c) = 0$. If $f''(c) > 0$, f has a local minimum at c ; if $f''(c) < 0$, a local maximum. If $f''(c) = 0$, the test is inconclusive.

Proof. If $f''(c) > 0$, continuity keeps $f'' > 0$ near c , so f' increases through $f'(c) = 0$: negative before, positive after. The first derivative test applies. ■

The inconclusive case is genuine: x^4 , $-x^4$, and x^3 all have $f'(0) = f''(0) = 0$, and have a minimum, a maximum, and neither.

Example 5.19 (Concavity and inflection). For $f(x) = x^4 - 4x^3$, find the intervals of concavity, the inflection points, and the local extrema.

Solution. $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$ and $f''(x) = 12x^2 - 24x = 12x(x - 2)$. Critical numbers: 0 and 3. At 3, $f''(3) = 36 > 0$: a local minimum $f(3) = -27$. At 0, $f''(0) = 0$ and the second test fails; but $f' < 0$ on both sides of 0 (the factor x^2 does not change sign), so there is no extremum. Concavity: $f'' > 0$ for $x < 0$ and $x > 2$ (up), $f'' < 0$ on $(0, 2)$ (down). Inflection points at $(0, 0)$ and $(2, -16)$. At $(0, 0)$ the tangent is horizontal and crosses the graph: a horizontal inflection.

COMMON ERROR. $f''(c) = 0$ does not make c an inflection point: x^4 has $f''(0) = 0$ but is concave upward on both sides. Concavity must actually change. Conversely, an inflection can occur where f'' fails to exist, as for $x^{1/3}$ at 0.

5.5 CURVE SKETCHING

A computer plots a graph in an instant, but it plots only the window it is given, and it can miss a vertical asymptote between two sample points, a tiny local minimum, or behavior far away. Calculus tells us which features exist and where, so that we know what a correct picture must show. A checklist:

- A. *Domain and intercepts.*
- B. *Symmetry:* even, odd, or periodic.
- C. *Asymptotes:* vertical (where the function is unbounded), horizontal or slant (as $x \rightarrow \pm\infty$).
- D. *Intervals of increase and decrease,* from the sign of f' , *and local extrema.*
- E. *Concavity and inflection points,* from the sign of f'' .
- F. *Sketch,* plotting the special points first.

Example 5.20 (A rational function). Sketch $f(x) = \frac{2x^2}{x^2 - 1}$.

Solution. A. Domain $x \neq \pm 1$; the only intercept is $(0, 0)$. B. Even. C. Vertical asymptotes $x = \pm 1$: as $x \rightarrow 1^+$, $f \rightarrow +\infty$, and as $x \rightarrow 1^-$, $f \rightarrow -\infty$ (the numerator is near 2, the denominator near 0 with the sign of $x - 1$); symmetrically at -1 . Horizontal asymptote $y = 2$ as $x \rightarrow \pm\infty$.

D. $f'(x) = \frac{4x(x^2-1)-2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$. So $f' > 0$ for $x < 0$ ($x \neq -1$) and $f' < 0$ for $x > 0$ ($x \neq 1$): f increases on $(-\infty, -1)$ and $(-1, 0)$ and decreases on $(0, 1)$ and $(1, \infty)$. Local maximum $f(0) = 0$.

E. $f''(x) = \frac{12x^2+4}{(x^2-1)^3}$. The numerator is always positive, so f'' has the sign of $(x^2 - 1)^3$: concave upward for $|x| > 1$, downward for $|x| < 1$. No inflection points, since ± 1 are not in the domain.

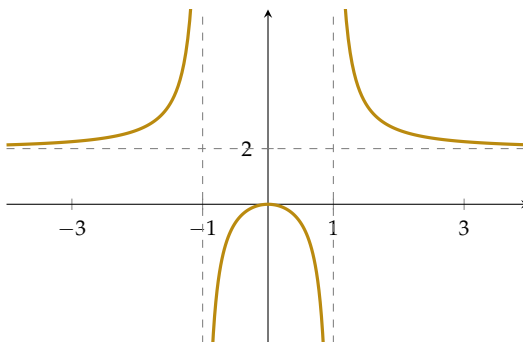


Figure 5.2: $y = \frac{2x^2}{x^2-1}$, with asymptotes $x = \pm 1$ and $y = 2$.

Example 5.21 (A function with a cusp). Sketch $g(x) = x^{2/3}(x - 5)$ on $\mathbb{R}$.

Solution. From Example 5.5, $g'(x) = \frac{5(x-2)}{3x^{1/3}}$. Intercepts $(0, 0)$ and $(5, 0)$. No symmetry, no asymptotes; $g \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$.

Sign of g' : for $x < 0$, numerator and denominator are both negative, $g' > 0$; on $(0, 2)$, $g' < 0$; for $x > 2$, $g' > 0$. Local maximum $g(0) = 0$, at a cusp (since $g' \rightarrow +\infty$ from the left and $-\infty$ from the right), and local minimum $g(2) = -3\sqrt[3]{4} \approx -4.76$.

$g''(x) = \frac{10(x+1)}{9x^{4/3}}$. The denominator is positive, so $g'' < 0$ for $x < -1$ and $g'' > 0$ for $x > -1$, $x \neq 0$. Inflection point at $(-1, -6)$.

Example 5.22 (An exponential). Sketch $h(x) = xe^{-x}$.

Solution. Domain $\mathbb{R}$; intercept $(0, 0)$. As $x \rightarrow -\infty$, $h \rightarrow -\infty$. As $x \rightarrow \infty$, $h \rightarrow 0$, because the exponential decays faster than x grows (proved in Chapter 6 with L'Hôpital's rule), so $y = 0$ is a horizontal asymptote on the right.

$h'(x) = (1 - x)e^{-x}$: increasing for $x < 1$, decreasing for $x > 1$, with absolute maximum $h(1) = \frac{1}{e} \approx 0.368$.

$h''(x) = (x - 2)e^{-x}$: concave down for $x < 2$, up for $x > 2$, inflection at $(2, \frac{2}{e^2}) \approx (2, 0.271)$. The function xe^{-x} , and its relatives $x^n e^{-x}$, describe the shape of radial probability densities in atoms (Volume IV): the maximum at $x = n$ is the most probable distance.

5.5.1 SLANT ASYMPTOTES AND THE SHAPE AT INFINITY

If $f(x) - (mx + b) \rightarrow 0$ as $x \rightarrow \infty$, the line $y = mx + b$ is a slant asymptote. For rational functions it is found by division, as in Chapter 2. For others, $m = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$ and $b = \lim_{x \rightarrow \infty} (f(x) - mx)$. For example, $\sqrt{x^2 + 2x}$ has $m = 1$ and $b = \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x) = 1$, so $y = x + 1$ is an asymptote as $x \rightarrow \infty$, and $y = -x - 1$ as $x \rightarrow -\infty$.

5.6 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— **Problem 5.1** An inequality by monotonicity

Prove that $e^x \geq 1 + x$ for all x , and that $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$.

Solution. Let $f(x) = e^x - 1 - x$. Then $f'(x) = e^x - 1$, negative for $x < 0$ and positive for $x > 0$. So f decreases to its minimum $f(0) = 0$ and then increases: $f \geq 0$ everywhere, with equality only at 0. Geometrically, the tangent line $y = 1 + x$ at $(0, 1)$ lies below the graph of the convex function e^x .

For the second, let $g(x) = x - \sin x$: $g(0) = 0$ and $g'(x) = 1 - \cos x > 0$ on $(0, \frac{\pi}{2})$, so g increases and $g(x) > 0$. Let $k(x) = \tan x - x$: $k(0) = 0$ and $k'(x) = \sec^2 x - 1 = \tan^2 x > 0$, so $k(x) > 0$. The method, comparing a function with its value at an endpoint via the sign of the derivative, proves a vast number of inequalities.

— **Problem 5.2** Where the MVT point is

For $f(x) = \sqrt{x}$ on $[0, 4]$, find all c satisfying the conclusion of the Mean Value Theorem. Then show that for $f(x) = x^2$ on any interval $[a, b]$, c is the midpoint.

Solution. The secant slope is $\frac{2-0}{4} = \frac{1}{2}$, and $f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{2}$ gives $c = 1$. Note that f is not differentiable at 0, but the theorem requires differentiability only on the open interval, and continuity on the closed one.

For x^2 : $\frac{b^2-a^2}{b-a} = a+b = 2c$, so $c = \frac{a+b}{2}$. For a quadratic, the tangent at the midpoint is always parallel to the chord, a property known to Archimedes, who used it to find the area of a parabolic segment.

— **Problem 5.3** Exactly three roots

Prove that $x^5 - 3x + 1 = 0$ has exactly three real roots, completing the worked problem of Chapter 2.

Solution. The IVT gave at least three. Suppose there were four or more. Applying Rolle's theorem between consecutive roots would give at least three zeros of $f'(x) = 5x^4 - 3$. But $5x^4 = 3$ has only two real solutions, $x = \pm(3/5)^{1/4}$. Hence there are exactly three roots. The same counting shows a polynomial with k distinct real critical points has at most $k + 1$ real roots.

— **Problem 5.4** A full sketch

Sketch $f(x) = \frac{x^2}{x-1}$, including the slant asymptote.

Solution. Domain $x \neq 1$; intercept $(0, 0)$. Division gives $f(x) = x + 1 + \frac{1}{x-1}$, so $y = x + 1$ is a slant asymptote, approached from above for $x > 1$ and from below for $x < 1$. Vertical asymptote $x = 1$: $f \rightarrow +\infty$ as $x \rightarrow 1^+$ and $-\infty$ as $x \rightarrow 1^-$.

$f'(x) = 1 - \frac{1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$. Critical numbers 0 and 2. f increases on $(-\infty, 0)$ and $(2, \infty)$, decreases on $(0, 1)$ and $(1, 2)$. Local maximum $f(0) = 0$, local minimum $f(2) = 4$. Note that the local maximum is smaller than the local minimum: they lie on different branches.

$f''(x) = \frac{2}{(x-1)^3}$: concave down for $x < 1$, up for $x > 1$, with no inflection point. The graph is a hyperbola, whose asymptotes are $x = 1$ and $y = x + 1$.

— **Problem 5.5** A family of curves

For which values of k does $f(x) = x^3 + kx^2 + 3x$ have local extrema? For which values does it have an inflection point, and where?

Solution. $f'(x) = 3x^2 + 2kx + 3$, a quadratic with discriminant $4k^2 - 36$. If $|k| > 3$, f' has two distinct real roots and changes sign at each, giving a local maximum and a local minimum. If $|k| \leq 3$, $f' \geq 0$ everywhere (with equality at most at one point), f is increasing, and there are no extrema. The borderline $k = \pm 3$ gives $f'(x) = 3(x \pm 1)^2$, which touches zero without changing sign.

$f''(x) = 6x + 2k$ changes sign at $x = -\frac{k}{3}$ for every k : every cubic has exactly one inflection point. It is also the center of symmetry of the cubic, and when there are two extrema, their midpoint.

— **Problem 5.6** A function with $f'(0) > 0$ that is not increasing near 0

Let $f(x) = x + 2x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) = 1$ but that f is not increasing on any interval containing 0.

Solution. $f'(0) = \lim_{h \rightarrow 0} \frac{h + 2h^2 \sin(1/h)}{h} = \lim_{h \rightarrow 0} (1 + 2h \sin \frac{1}{h}) = 1$, by squeezing. For $x \neq 0$, $f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$. At $x = \frac{1}{2\pi n}$, $\sin \frac{1}{x} = 0$ and $\cos \frac{1}{x} = 1$, so $f'(x) = -1 < 0$. These points accumulate at 0, and near each f is decreasing. So no interval around 0 is one on which f increases. The increasing test needs $f' > 0$ on an interval, not at a point. A positive derivative at a single point only guarantees $f(x) > f(0)$ for small $x > 0$ and $f(x) < f(0)$ for small $x < 0$.

— **Problem 5.7** Bounding a value with the MVT

If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ be? If also $f'(x) \leq 5$, find the range of possible values of $f(4)$.

Solution. By the MVT, $f(4) - f(1) = 3f'(c)$ for some $c \in (1, 4)$. Since $f'(c) \geq 2$, $f(4) \geq 10 + 6 = 16$. With $2 \leq f' \leq 5$, $16 \leq f(4) \leq 25$. Both bounds are attained, by the linear functions with slopes 2 and 5. This is the logic of speed cameras on toll roads: if your average speed between two cameras exceeded the limit, the MVT guarantees that at some instant you were speeding.

EXERCISES

A. COMPUTATION

5.1 Find the critical numbers of $f(x) = x^3 + 6x^2 - 15x$ and of $g(x) = x^{4/5}(x - 4)^2$.

5.2 Find the critical numbers of $f(\theta) = 2 \cos \theta + \sin^2 \theta$ on $[0, 2\pi]$.

5.3 Find the absolute extrema of $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1, 4]$.

- 5.4** Find the absolute extrema of $f(x) = \frac{x}{x^2 + 4}$ on $[0, 3]$.
- 5.5** Find the absolute extrema of $f(x) = x - 2 \ln x$ on $[\frac{1}{2}, 4]$.
- 5.6** Find the absolute extrema of $f(t) = t\sqrt{4 - t^2}$ on $[-1, 2]$.
- 5.7** Verify Rolle's theorem for $f(x) = x^3 - x^2 - 6x + 2$ on $[0, 3]$.
- 5.8** Find all c satisfying the MVT for $f(x) = x^3 - 3x + 2$ on $[-2, 2]$.
- 5.9** Find all c satisfying the MVT for $f(x) = \ln x$ on $[1, e]$.
- 5.10** Show that $x^3 - 15x + c = 0$ has at most one root in $[-2, 2]$ for any c .
- 5.11** Find the intervals of increase and decrease and the local extrema of $f(x) = x^4 - 2x^2 + 3$.
- 5.12** Find the intervals of increase and decrease and the local extrema of $f(x) = x\sqrt{6 - x}$.
- 5.13** Find the intervals of increase and decrease of $f(x) = x^2e^{-x}$.
- 5.14** Find the intervals of concavity and the inflection points of $f(x) = x^3 - 6x^2 + 9x$.
- 5.15** Find the intervals of concavity and the inflection points of $f(x) = \ln(x^2 + 1)$.
- 5.16** Use the second derivative test to classify the critical points of $f(x) = 2x^3 + 3x^2 - 36x$.
- 5.17** Use the second derivative test, where possible, to classify the critical points of $f(x) = x^5 - 5x^4$.
- 5.18** Sketch $y = \frac{x}{x^2 + 1}$, giving extrema, inflection points, and asymptotes.
- 5.19** Sketch $y = \frac{x^2 - 4}{x^2 - 9}$.
- 5.20** Sketch $y = x^{1/3}(x + 4)$.
- 5.21** Sketch $y = x - 2 \sin x$ on $[0, 2\pi]$.
- 5.22** Sketch $y = e^{-x^2/2}$, the bell curve, and find its inflection points.
- 5.23** Sketch $y = \ln(\cos x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

- 5.24** Find the slant asymptote of $y = \frac{x^3}{x^2 + 1}$ and sketch.
- 5.25** Find constants a, b such that $f(x) = x^3 + ax^2 + bx$ has a local minimum at $x = 1$ with value -2 .
- 5.26** Sketch the graph of a continuous function with $f'(x) > 0$ for $|x| < 2$, $f'(x) < 0$ for $|x| > 2$, $f''(x) < 0$ for $x > 0$, and $f''(x) > 0$ for $x < 0$, $f(0) = 0$.
- 5.27** For $f(x) = x + \frac{1}{x}$, find the local extrema, and explain why the local maximum is less than the local minimum.
- 5.28** Find the local extrema of $f(x) = x^x$ for $x > 0$.
- 5.29** If $f(0) = -3$ and $f'(x) \leq 5$ for all x , how large can $f(2)$ be?
- 5.30** Show that $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$.

B. THEORY AND PROOF

- 5.31** Prove that if $f'(x) = f(x)$ for all x , then $f(x) = Ce^x$. (Consider $f(x)e^{-x}$.)
- 5.32** Prove that $|\arctan a - \arctan b| \leq |a - b|$.
- 5.33** Prove that a polynomial of degree n has at most $n - 1$ critical numbers and at most $n - 2$ inflection points.
- 5.34** Prove that if $f'' > 0$ on an interval, then f has at most one critical number there, and it is a minimum.
- 5.35** Show that if f is differentiable and $|f'(x)| \leq M < 1$ on $\mathbb{R}$, then f has at most one fixed point.
- 5.36** Prove that the inflection point of a cubic is its center of symmetry: $f(c + h) + f(c - h) = 2f(c)$.
- 5.37** Prove that if f is concave up on an interval, then $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$ for a, b in the interval.
- 5.38** Prove Bernoulli's inequality $(1+x)^n \geq 1+nx$ for $x > -1$ and integers $n \geq 1$, by calculus.
- 5.39** Use Cauchy's MVT to prove that if $f(0) = g(0) = 0$ and $g' \neq 0$ near 0, then $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)}$ when the latter exists.

C. CHALLENGE

- 5.40** Prove that a function with $f'(x)$ existing everywhere satisfies the intermediate value property for f' (Darboux's theorem), even though f' need not be continuous.
- 5.41** Show that $\frac{2}{\pi}x \leq \sin x \leq x$ for $0 \leq x \leq \frac{\pi}{2}$ (Jordan's inequality).
- 5.42** Find the largest value of a for which $a^x \geq x$ for all $x > 0$. (The answer involves $e^{1/e}$.)
- 5.43** Prove that if f is twice differentiable on $[0, 2]$ with $f(0) = f(1) = f(2) = 0$, then $f''(c) = 0$ for some c .
- 5.44** Prove that $e^\pi > \pi^e$ by studying $\frac{\ln x}{x}$.

CHAPTER 6

APPLICATIONS OF THE DERIVATIVE

LEARNING OBJECTIVES

- Prove L'Hôpital's rule and use it on all seven indeterminate forms.
- Compare growth rates of logarithms, powers, and exponentials.
- Model and solve optimization problems, and justify that the answer is a global extremum.
- Derive Newton's method, analyze its convergence, and recognize when it fails.
- Find antiderivatives and solve initial value problems for motion.

The Mean Value Theorem gave us the shape of a graph. This chapter puts that knowledge to work: to evaluate limits that algebra cannot crack, to find the best value of a quantity, to solve equations numerically, and to recover a function from its rate of change.

6.1 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

A limit of the form $\lim \frac{f}{g}$ with $f \rightarrow 0$ and $g \rightarrow 0$ is an *indeterminate form of type $\frac{0}{0}$* : the answer depends on how fast each tends to 0, and can be anything. We have met such limits as derivatives, $\frac{\sin x}{x}$, and $\frac{x^2-1}{x-1}$. L'Hôpital's rule replaces the ratio of functions by the ratio of their rates of change.

Theorem 6.1 (L'Hôpital's rule). *Suppose f and g are differentiable near a (except possibly at a), $g'(x) \neq 0$ near a , and either*

- $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, or
- $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$.

If $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ (a number, or $\pm\infty$), then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$. The same holds for one-sided limits and for $a = \pm\infty$.

Proof of the $\frac{0}{0}$ case, $x \rightarrow a^+$. Define $f(a) = g(a) = 0$, making both continuous on $[a, x]$. By Cauchy's Mean Value Theorem (Theorem 5.12) there is $c \in (a, x)$ with $f(x)g'(c) = g(x)f'(c)$. Since $g' \neq 0$, $g(x) \neq 0$ (by Rolle's theorem, as $g(a) = 0$), and so $\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$. As $x \rightarrow a^+$, $c \rightarrow a^+$, so the right side tends to L . The $\frac{\infty}{\infty}$ case and the case $a = \infty$ (substitute $x = \frac{1}{t}$) are proved in Appendix A. ■

Example 6.2 (Type $\frac{0}{0}$ and $\frac{\infty}{\infty}$). Evaluate (a) $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$, (b)

$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$, (c) $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$, (d) $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$.

Solution. (a) $\frac{0}{0}$; the rule twice: $\frac{e^x - 1}{2x} \rightarrow \frac{e^x}{2} \rightarrow \frac{1}{2}$.

(b) $\frac{0}{0}$, three times: $\frac{1 - \cos x}{3x^2} \rightarrow \frac{\sin x}{6x} \rightarrow \frac{\cos x}{6} \rightarrow \frac{1}{6}$.

(c) $\frac{\infty}{\infty}$: $\frac{1/x}{1/(2\sqrt{x})} = \frac{2}{\sqrt{x}} \rightarrow 0$.

(d) $\frac{\infty}{\infty}$, three times: $\frac{3x^2}{e^x} \rightarrow \frac{6x}{e^x} \rightarrow \frac{6}{e^x} \rightarrow 0$.

COMMON ERROR. Applying the rule when the form is not indeterminate. $\lim_{x \rightarrow 0} \frac{\cos x}{x+1} = 1$, but differentiating gives $\frac{-\sin x}{1} \rightarrow 0$. Check the form at every application. And the rule uses the quotient of derivatives $\frac{f'}{g'}$, not the derivative of the quotient $(\frac{f}{g})'$.

Theorem 6.3 (Growth rates). For every $p > 0$ and $b > 1$: $\lim_{x \rightarrow \infty} \frac{\ln x}{x^p} = 0$ and $\lim_{x \rightarrow \infty} \frac{x^p}{b^x} = 0$.

Proof. $\frac{\ln x}{x^p}$: by the rule, $\frac{1/x}{px^{p-1}} = \frac{1}{px^p} \rightarrow 0$. For $\frac{x^p}{b^x}$, apply the rule $\lceil p \rceil$ times: each application lowers the power of x by one and multiplies the denominator by $\ln b > 0$, until the numerator's power is ≤ 0 and the quotient tends to 0. ■

Logarithms grow more slowly than any power, and powers more slowly than any exponential. In computer science, an algorithm taking $n \ln n$ steps beats one taking $n^{1.01}$ for large enough n , and any polynomial-time algorithm beats an exponential one eventually.

6.1.1 OTHER INDETERMINATE FORMS

The forms $0 \cdot \infty$ and $\infty - \infty$ are rewritten as quotients. The forms 0^0 , ∞^0 , and 1^∞ arise from f^g ; take logarithms, since $\ln(f^g) = g \ln f$ is of type $0 \cdot \infty$.

Example 6.4 (Products, differences, and powers). Evaluate (a) $\lim_{x \rightarrow 0^+} x \ln x$,

(b) $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$, (c) $\lim_{x \rightarrow 0^+} x^x$, (d) $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^x$.

Solution. (a) $0 \cdot (-\infty)$. Write $\frac{\ln x}{1/x}$, type $\frac{-\infty}{-\infty}$: $\frac{1/x}{-1/x^2} = -x \rightarrow 0$.

(b) $\infty - \infty$. Combine: $\frac{e^x - 1 - x}{x(e^x - 1)}$, type $\frac{0}{0}$. Twice: $\frac{e^x - 1}{e^x - 1 + xe^x} \rightarrow \frac{e^x}{2e^x + xe^x} \rightarrow \frac{1}{2}$.

(c) 0^0 . $\ln(x^x) = x \ln x \rightarrow 0$ by (a), so $x^x \rightarrow e^0 = 1$.

(d) 1^∞ . $\ln y = x \ln \left(1 + \frac{a}{x} \right) = \frac{\ln(1+a/x)}{1/x}$, type $\frac{0}{0}$. Differentiating: $\frac{\frac{-a/x^2}{1+a/x}}{-1/x^2} = \frac{a}{1+a/x} \rightarrow a$. So $y \rightarrow e^a$, recovering Theorem 4.14 for continuous compounding.

Example 6.5 (When the rule fails to help). $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = 1$, as dividing by x shows. The rule gives $\frac{1 + \cos x}{1}$, which has no limit. The rule says that if $\frac{f'}{g'}$ has a limit then so does $\frac{f}{g}$, not the converse. And $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x}$ cycles: the rule gives $\frac{x}{\sqrt{x^2+1}}$, then back again. Dividing by x gives 1 at once.

6.2 OPTIMIZATION

Optimization problems ask for the best: the largest area, the least cost, the shortest time. The procedure:

1. Understand the problem; draw a diagram; name the quantity Q to be optimized and the variables.
2. Express Q as a function of one variable, using the constraints to eliminate the others.
3. Determine the domain of that function from the physical situation.
4. Find the critical numbers, and determine the absolute extremum: by the closed interval method on a closed bounded domain, or by the first derivative test (a single critical number at which f' changes sign gives the absolute extremum on an interval).
5. Answer the question that was asked, with units, and check that the answer is sensible.

Example 6.6 (An open box). Squares of side x are cut from the corners of a 12×20 cm sheet, and the sides folded up to make an open box. Find the largest volume.

Solution. $V(x) = x(12 - 2x)(20 - 2x) = 4x^3 - 64x^2 + 240x$ on $[0, 6]$.
 $V'(x) = 12x^2 - 128x + 240 = 4(3x^2 - 32x + 60)$, with roots $x = \frac{32 \pm \sqrt{1024 - 720}}{6} = \frac{16 \pm 2\sqrt{19}}{3}$. Only $x = \frac{16 - 2\sqrt{19}}{3} \approx 2.427$ lies in $[0, 6]$. At the endpoints $V = 0$, so the maximum is $V(2.427) \approx 262.7 \text{ cm}^3$.

Example 6.7 (The cheapest can). A cylindrical can must hold 500 cm^3 . The top and bottom cost twice as much per unit area as the side. What dimensions minimize the cost?

Solution. With radius r and height h , $\pi r^2 h = 500$, so $h = \frac{500}{\pi r^2}$. The cost in units of the side's price per cm^2 is $C = 2\pi r h + 2 \cdot 2\pi r^2 = \frac{1000}{r} + 4\pi r^2$, for $r > 0$. Then $C'(r) = -\frac{1000}{r^2} + 8\pi r = 0$ gives $8\pi r^3 = 1000$, so $\pi r^3 = 125$ and $r = \frac{5}{\pi^{1/3}} \approx 3.414 \text{ cm}$. Since $500 = 4\pi r^3$ at this radius, $h = \frac{500}{\pi r^2} = \frac{4\pi r^3}{\pi r^2} = 4r \approx 13.66 \text{ cm}$. Since $C' < 0$ for smaller r and $C' > 0$ for larger, this is the absolute minimum. The optimal height is four times the radius, twice the diameter. If all surfaces cost the same, the same calculation gives $h = 2r$: height equal to diameter.

Example 6.8 (Snell's law from least time). Light travels from point A in air (speed v_1) to point B in water (speed v_2), crossing a flat surface. Show that the path of least time satisfies $\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$, where θ_1, θ_2 are the angles from the normal.

Solution. Put the surface on the x -axis, $A = (0, a)$ and $B = (d, -b)$, and let the light cross at $(x, 0)$. The time is

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d - x)^2}}{v_2},$$

$$T'(x) = \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{d - x}{v_2 \sqrt{b^2 + (d - x)^2}} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}.$$

$T'(0) < 0 < T'(d)$, and $T'' > 0$ (each term of T is convex), so T has a single critical point, a minimum, where $T' = 0$. That is Snell's law of refraction, derived by Fermat in 1662 from his principle that light takes the path of least time.

Example 6.9 (The nearest point). Find the point on the parabola $y^2 = 2x$ closest to $(1, 4)$.

Solution. Minimize the square of the distance, which has the same minimizer and no square root. With $x = \frac{y^2}{2}$: $D(y) = (\frac{y^2}{2} - 1)^2 + (y - 4)^2$. $D'(y) = 2(\frac{y^2}{2} - 1)y + 2(y - 4) = y^3 - 8$, zero at $y = 2$. $D' < 0$ for $y < 2$ and > 0 for $y > 2$: an absolute minimum. The point is $(2, 2)$, at distance $\sqrt{5}$. At that point the segment to $(1, 4)$ is perpendicular to the tangent: the slope of the segment is -2 and the tangent slope is $\frac{1}{y} = \frac{1}{2}$.

6.2.1 OPTIMIZATION IN ECONOMICS

If $C(q)$ is the cost and $R(q) = qp(q)$ the revenue from selling q units at price $p(q)$, the profit is $P = R - C$, and $P'(q) = 0$ gives $R'(q) = C'(q)$: *profit is maximized where marginal revenue equals marginal cost*. (The second derivative test requires $R'' < C''$ there.)

Example 6.10 (Pricing). A shop sells 200 units a week at \$30. Each \$1 price cut sells 20 more units. Units cost \$10 each. What price maximizes profit?

Solution. With a price cut of x dollars, the price is $30 - x$, sales are $200 + 20x$, and the profit is $P(x) = (30 - x - 10)(200 + 20x) = (20 - x)(200 + 20x)$. Then $P'(x) = -(200 + 20x) + 20(20 - x) = 200 - 40x$, zero at $x = 5$. $P'' = -40 < 0$, a maximum. The best price is \$25, selling 300 units for a profit of \$4500, up from \$4000.

6.3 NEWTON'S METHOD

Most equations cannot be solved exactly. Newton's method approximates a root of $f(x) = 0$ by repeatedly replacing the curve with its tangent line. From a guess x_n , the tangent $y = f(x_n) + f'(x_n)(x - x_n)$ crosses the axis at

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 6.11 (The square root of two). Apply Newton's method to $f(x) = x^2 - 2$ with $x_1 = 1$.

Solution. $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right)$, the average of x_n and $2/x_n$, the

Babylonian method known four thousand years ago. The iterates:

n	x_n
2	1.5
3	1.416 666 666 666 67
4	1.414 215 686 274 51
5	1.414 213 562 374 69
6	1.414 213 562 373 10

The number of correct digits roughly doubles at each step: 1, 3, 6, 12, then full machine precision.

Theorem 6.12 (Quadratic convergence). Suppose $f(r) = 0$, $f'(r) \neq 0$, and f'' is continuous near r . If x_n is close enough to r , then

$$|x_{n+1} - r| \leq K|x_n - r|^2, \quad K \approx \left| \frac{f''(r)}{2f'(r)} \right|.$$

Sketch. By Taylor's theorem (Chapter 13), $0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{1}{2}f''(\xi)(r - x_n)^2$ for some ξ between r and x_n . Dividing by $f'(x_n)$ and rearranging, $x_{n+1} - r = x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{f''(\xi)}{2f'(x_n)}(x_n - r)^2$. ■

If the error is 10^{-3} , the next is about $K \cdot 10^{-6}$, then $K^3 \cdot 10^{-12}$: the doubling of digits seen above.

COMMON ERROR. Newton's method can fail. A starting point where $f'(x_1) \approx 0$ throws the next iterate far away: for $x^3 - 2x - 5$ starting at $x_1 = 1$ (where $f'(1) = 1$), the iterates jump to 7 and then crawl back, 4.77, 3.35, 2.53, 2.17, while $x_1 = 2$ converges to 2.094 551 48 in three steps. For $f(x) = x^{1/3}$, the formula gives $x_{n+1} = -2x_n$ from any start: the iterates oscillate and diverge, because f' is infinite at the root. Always check that the iterates settle down.

6.4 ANTIDERIVATIVES

Definition 6.13 (Antiderivative). F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

By Theorem 5.8, if F is one antiderivative, every antiderivative on I is $F + C$ for a constant C . The family is the *general antiderivative*, written $\int f(x) dx = F(x) + C$; the reason for the integral sign appears in Chapter 7.

$f(x)$	antiderivative	$f(x)$	antiderivative
x^r ($r \neq -1$)	$\frac{x^{r+1}}{r+1}$	$\frac{1}{x}$	$\ln x $
e^{kx}	$\frac{1}{k} e^{kx}$	b^x	$\frac{b^x}{\ln b}$
$\cos kx$	$\frac{1}{k} \sin kx$	$\sin kx$	$-\frac{1}{k} \cos kx$
$\sec^2 x$	$\tan x$	$\sec x \tan x$	$\sec x$
$\frac{1}{1+x^2}$	$\arctan x$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\cosh x$	$\sinh x$	$\sinh x$	$\cosh x$

Each entry is checked by differentiating the right column. The antiderivative $\ln |x|$ is valid on each of the intervals $(-\infty, 0)$ and $(0, \infty)$ separately, with an independent constant on each.

Example 6.14 (An initial value problem). Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Solution. Antidifferentiate twice: $f'(x) = 4x^3 + 3x^2 - 4x + C$ and $f(x) = x^4 + x^3 - 2x^2 + Cx + D$. Then $f(0) = D = 4$, and $f(1) = 1 + 1 - 2 + C + 4 = 1$ gives $C = -3$. So $f(x) = x^4 + x^3 - 2x^2 - 3x + 4$. Two conditions fix the two constants of integration.

6.4.1 MOTION WITH GIVEN ACCELERATION

If $a(t)$ is known, v is an antiderivative of a , and s an antiderivative of v . For constant acceleration a :

$$v(t) = v_0 + at, \quad s(t) = s_0 + v_0 t + \frac{1}{2} at^2,$$

the formulas of introductory physics, derived in two lines.

Example 6.15 (Braking distance). A car traveling at 30 m/s brakes with constant deceleration and stops in 75 m. What is the deceleration, and how long does it take to stop?

Solution. With $a = -k$: $v = 30 - kt$, $s = 30t - \frac{1}{2}kt^2$. It stops when $v = 0$, at $t = \frac{30}{k}$, having traveled $\frac{900}{k} - \frac{450}{k} = \frac{450}{k} = 75$. So $k = 6 \text{ m/s}^2$, and it stops in 5 s. The stopping distance is proportional to the square of the speed: at 60 m/s it would be 300 m, four times as far.

6.5 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— **Problem 6.1** A limit needing logarithms and L'Hôpital

Evaluate $\lim_{x \rightarrow 0^+} (\sin x)^{\tan x}$ and $\lim_{x \rightarrow \infty} (x + e^x)^{1/x}$.

Solution. First, type 0^0 . $\ln y = \tan x \ln \sin x = \frac{\ln \sin x}{\cot x}$, type $\frac{-\infty}{\infty}$. Differentiating, $\frac{\cos x / \sin x}{-\csc^2 x} = -\sin x \cos x \rightarrow 0$. So $y \rightarrow e^0 = 1$.

Second, type ∞^0 . $\ln y = \frac{\ln(x+e^x)}{x}$, type $\frac{\infty}{\infty}$: $\frac{(1+e^x)/(x+e^x)}{1} = \frac{1+e^x}{x+e^x}$, again $\frac{\infty}{\infty}$: $\frac{e^x}{1+e^x} \rightarrow 1$. So $y \rightarrow e$. Intuitively, $x + e^x \approx e^x$ for large x , and $(e^x)^{1/x} = e$.

— **Problem 6.2** The strongest beam

The strength of a rectangular beam is proportional to its width times the square of its depth. Find the strongest beam that can be cut from a log of diameter d .

Solution. A rectangle of width w and depth h inscribed in a circle of diameter d has $w^2 + h^2 = d^2$. Maximize $S = wh^2 = w(d^2 - w^2)$ for $0 < w < d$: $S' = d^2 - 3w^2 = 0$ gives $w = \frac{d}{\sqrt{3}}$, and $h = \sqrt{d^2 - \frac{d^2}{3}} = d\sqrt{\frac{2}{3}}$. Since $S \rightarrow 0$ at both ends of the interval and there is one critical number, it is the maximum. The ratio of depth to width is $\sqrt{2} \approx 1.41$, the proportion still used for timber beams.

— **Problem 6.3** Running and swimming

A woman on the shore at A wants to reach a point B on the opposite bank of a straight river 3 km wide, 8 km downstream. She runs at 8 km/h and swims at 6 km/h (ignore the current). Where should she enter the water?

Solution. Let her run x km along the bank, then swim straight to B , a distance

$\sqrt{9 + (8 - x)^2}$. The time is $T(x) = \frac{x}{8} + \frac{\sqrt{9 + (8 - x)^2}}{6}$ on $[0, 8]$. Then

$$T'(x) = \frac{1}{8} - \frac{8 - x}{6\sqrt{9 + (8 - x)^2}} = 0.$$

With $u = 8 - x$, this says $6\sqrt{9 + u^2} = 8u$, so $36(9 + u^2) = 64u^2$, so $u^2 = \frac{324}{28}$, $u = \frac{9}{\sqrt{7}} \approx 3.402$, and $x \approx 4.598$ km. Comparing: $T(0) = \frac{\sqrt{73}}{6} \approx 1.424$ h, $T(8) = 1.5$ h, $T(4.598) = 0.575 + 0.757 = 1.331$ h. She should run about 4.6 km and then swim. The condition $\frac{u}{\sqrt{9 + u^2}} = \frac{6}{8}$ is Snell's law again (Example 6.8): the sine of the swimming angle equals the ratio of speeds.

— Problem 6.4 Newton's method for an intersection

Find, to six decimal places, the x -coordinate where $y = \cos x$ meets $y = x$.

Solution. Solve $f(x) = \cos x - x = 0$, with $f'(x) = -\sin x - 1$. From the graphs the root is near 0.7; start at $x_1 = 0.5$: $x_2 = 0.755222$, $x_3 = 0.739142$, $x_4 = 0.73908513$, $x_5 = 0.73908513$. The root is 0.739085, confirming the bisection estimate of Chapter 2, which needed about twenty steps for the same accuracy. This number is the *Dottie number*: pressing the cosine key on a calculator repeatedly (in radians) converges to it, from any start, by the fixed-point iteration $x_{n+1} = \cos x_n$, which converges only linearly.

— Problem 6.5 Comparing growth precisely

For which $x > 0$ is $2^x > x^{10}$? Show that the inequality holds for all $x > 58.8$ but fails at $x = 10$, so that “eventually” can be far away.

Solution. Compare logarithms: $2^x > x^{10}$ iff $x \ln 2 > 10 \ln x$, iff $g(x) = \frac{\ln x}{x} < \frac{\ln 2}{10} = 0.0693$. The function $\frac{\ln x}{x}$ increases on $(0, e)$ and decreases on (e, ∞) , with maximum $\frac{1}{e} = 0.368$, so the equation $g(x) = 0.0693$ has two solutions: near $x = 1.077$ and $x = 58.77$ (by Newton's method). Hence $2^x > x^{10}$ for $x < 1.077$ and for $x > 58.77$, but not in between: at $x = 10$, $2^{10} = 1024$ while 10^{10} is ten billion. The exponential eventually wins (Theorem 6.3), but not until $x \approx 58.8$, where both sides exceed 10^{17} .

— Problem 6.6 A ladder around a corner

A hallway 2 m wide meets, at a right angle, a hallway 3 m wide. What is the longest ladder that can be carried horizontally around the corner?

Solution. A ladder that fits must fit along every line touching the inner corner and the two outer walls. Such a line making angle θ with the wider hallway's

wall has length $L(\theta) = \frac{3}{\sin \theta} + \frac{2}{\cos \theta}$, $0 < \theta < \frac{\pi}{2}$. The longest ladder that fits is the *minimum* of L . Then

$$L'(\theta) = -\frac{3 \cos \theta}{\sin^2 \theta} + \frac{2 \sin \theta}{\cos^2 \theta} = 0 \implies \tan^3 \theta = \frac{3}{2}, \quad \tan \theta = (1.5)^{1/3}.$$

In general, for widths a, b , $\tan \theta = (a/b)^{1/3}$ and $L_{\min} = (a^{2/3} + b^{2/3})^{3/2}$. Here $L_{\min} = (3^{2/3} + 2^{2/3})^{3/2} \approx 7.02$ m. Since $L \rightarrow \infty$ at both ends, the critical point is the minimum. A minimum of one quantity gives a maximum of another: the typical twist of a well-posed optimization problem.

— Problem 6.7 Recovering motion

A particle has acceleration $a(t) = 6t - 4$ m/s², initial velocity $v(0) = -3$ m/s, and initial position $s(0) = 2$ m. Find $s(t)$, the times at which it is at rest, and the distance traveled from $t = 0$ to $t = 3$.

Solution. $v(t) = 3t^2 - 4t - 3$ and $s(t) = t^3 - 2t^2 - 3t + 2$. It is at rest when $3t^2 - 4t - 3 = 0$: $t = \frac{4+\sqrt{52}}{6} = \frac{2+\sqrt{13}}{3} \approx 1.869$ s (the other root is negative). Positions: $s(0) = 2$, $s(1.869) \approx -4.06$, $s(3) = 2$. The particle moves backward to -4.06 and then forward again, so the distance is $6.06 + 6.06 \approx 12.13$ m, while the displacement is 0. Distance is the integral of speed, not of velocity, a distinction made precise in Chapter 7.

EXERCISES

A. COMPUTATION

6.1 Evaluate $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$.

6.2 Evaluate $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$ and $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin x}$.

6.3 Evaluate $\lim_{x \rightarrow \infty} \frac{(\ln x)^3}{x}$ and $\lim_{x \rightarrow \infty} \frac{x^{100}}{e^{x/100}}$.

6.4 Evaluate $\lim_{x \rightarrow 0} \frac{\arctan x - x}{x^3}$.

6.5 Evaluate $\lim_{x \rightarrow 0^+} \sqrt{x} \ln x$ and $\lim_{x \rightarrow \infty} x e^{-x^2}$.

6.6 Evaluate $\lim_{x \rightarrow \pi/2^-} (\sec x - \tan x)$.

- 6.7** Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.
- 6.8** Evaluate $\lim_{x \rightarrow 0^+} x^{\sin x}$ and $\lim_{x \rightarrow \infty} x^{1/x}$.
- 6.9** Evaluate $\lim_{x \rightarrow 0} (1 + 2x)^{1/x}$ and $\lim_{x \rightarrow 0} (\cos x)^{1/x^2}$.
- 6.10** Evaluate $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1} \right)^x$.
- 6.11** Find two positive numbers with sum 20 whose product is as large as possible.
- 6.12** Find the dimensions of the rectangle of largest area inscribed in a semicircle of radius r .
- 6.13** A farmer has 600 m of fencing to enclose a rectangular field along a straight river, needing no fence along the river. Find the largest area.
- 6.14** A box with a square base and open top must have volume $32\,000 \text{ cm}^3$. Find the dimensions that minimize the material used.
- 6.15** Find the point on the line $y = 2x + 3$ closest to the origin.
- 6.16** Find the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$.
- 6.17** A cone is inscribed in a sphere of radius R . Find the largest volume of the cone.
- 6.18** A window is a rectangle topped by a semicircle. Its perimeter is 10 m. Find the dimensions that admit the most light.
- 6.19** A demand function is $p = 60 - 0.01q$ dollars and the cost is $C(q) = 2000 + 10q$. Find the production level and price that maximize profit.
- 6.20** A rectangular poster has 384 cm^2 of print, with margins of 4 cm top and bottom and 3 cm on the sides. Find the dimensions of the smallest poster.
- 6.21** Use Newton's method with $x_1 = 1$ to approximate $\sqrt[3]{5}$ to six decimal places.
- 6.22** Use Newton's method to find all roots of $x^4 - 2x^3 + 5x^2 - 6 = 0$ to six decimal places.
- 6.23** Use Newton's method to solve $e^{-x} = x$.

- 6.24** Show that Newton's method for $f(x) = \frac{1}{x} - a$ gives $x_{n+1} = x_n(2 - ax_n)$, which computes $\frac{1}{a}$ without division. Compute $\frac{1}{7}$ from $x_1 = 0.1$.
- 6.25** Find the general antiderivative of $f(x) = 6x^2 - 8x + 3$ and of $g(x) = \frac{2}{x} + 3e^x - \sec^2 x$.
- 6.26** Find the general antiderivative of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ and of $g(t) = \frac{t^3 + 2t}{t^2}$.
- 6.27** Find f if $f'(x) = 4 \cos x - 3 \sin x$ and $f(0) = 2$.
- 6.28** Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, $f(2) = 0$.
- 6.29** A stone is thrown down from a 450 m cliff at 5 m/s. Using $g = 9.8 \text{ m/s}^2$, when does it hit the ground, and how fast?
- 6.30** A car accelerates uniformly from 0 to 100 km/h in 8 s. Find the acceleration in m/s^2 and the distance covered.

B. THEORY AND PROOF

- 6.31** Prove that $\lim_{x \rightarrow \infty} \frac{(\ln x)^a}{x^b} = 0$ for all $a, b > 0$.
- 6.32** Show that $\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^n} = 0$ for every $n \geq 0$, and deduce that $f(x) = e^{-1/x^2}$ ($f(0) = 0$) has $f^{(n)}(0) = 0$ for all n .
- 6.33** Prove that among all rectangles with a given perimeter, the square has the largest area, and among those with a given area, the square has the smallest perimeter.
- 6.34** Prove that the point on a line $ax + by = c$ closest to the origin lies on the perpendicular from the origin, and derive the distance $\frac{|c|}{\sqrt{a^2 + b^2}}$.
- 6.35** Show that the reflection law (angle of incidence equals angle of reflection) follows from minimizing the path length from A to a mirror to B .
- 6.36** Explain why Newton's method applied to $f(x) = x^2 + 1$ cannot converge, and describe what the iterates do.
- 6.37** Show that Newton's method for $x^2 - a$ ($a > 0$) converges to $\sqrt{a}$ from every $x_1 > 0$. (Show $x_n \geq \sqrt{a}$ for $n \geq 2$ and the sequence decreases.)
- 6.38** Prove that if F and G are antiderivatives of f on an interval, they differ by a constant, and give a counterexample on a union of two intervals.

C. CHALLENGE

- 6.39** Find $\lim_{x \rightarrow 0} \frac{\sin(\tan x) - \tan(\sin x)}{x^7}$. (The answer is $-\frac{1}{30}$; computer algebra or the series of Chapter 13 help.)
- 6.40** A painting of height h hangs with its bottom edge a above eye level. Show that the viewing distance maximizing the subtended angle is $\sqrt{a(a+h)}$.
- 6.41** Find the distance from the point $(0, a)$ to the parabola $y = x^2$, for every $a > 0$. (The answer changes form at $a = \frac{1}{2}$.)
- 6.42** A rope of length L is cut into two pieces, one bent into a square and the other into a circle. How should it be cut to minimize, and to maximize, the total area?
- 6.43** Show that for $f(x) = x^2 - a$, the error $e_n = x_n - \sqrt{a}$ of Newton's method satisfies $e_{n+1} = \frac{e_n^2}{2x_n}$ exactly, and deduce the doubling of correct digits seen in Example 6.11.



PART II

INTEGRAL CALCULUS

Area and accumulation, the Fundamental Theorem of Calculus, techniques and applications of integration, and differential equations.

CHAPTER 7

THE DEFINITE INTEGRAL

LEARNING OBJECTIVES

- Compute areas and distances as limits of sums, using sigma notation and the formulas for $\sum i^k$.
- Define the definite integral by Riemann sums, and use its properties, including the comparison and mean value properties.
- State and prove both parts of the Fundamental Theorem of Calculus, and use it to evaluate integrals and net change.
- Evaluate indefinite and definite integrals by substitution, and exploit symmetry.
- Define the natural logarithm as an integral, derive its properties, and complete the theory of e^x .

Differential calculus asks for the rate of change of a quantity. Integral calculus asks the reverse: given the rate, how much has accumulated? Geometrically, the first question is about slopes of tangent lines, and the second about areas under curves. The two look unrelated, and for two thousand years they were treated separately: Archimedes computed areas by exhaustion in the third century BC, while tangents waited for the seventeenth century. The discovery that they are inverse operations, the Fundamental Theorem of Calculus, is the central fact of the subject, and the reason it is called *the* calculus.

7.1 AREA AND DISTANCE

7.1.1 THE AREA PROBLEM

What is the area of the region S under $y = x^2$ from 0 to 1? We know areas of rectangles, so we approximate S by rectangles. Divide $[0, 1]$ into n strips of width $\frac{1}{n}$, and on the i th strip $[\frac{i-1}{n}, \frac{i}{n}]$ erect a rectangle whose height is the value

at the right endpoint, $\left(\frac{i}{n}\right)^2$. The total area of these rectangles is

$$R_n = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2},$$

using a formula proved below. Since x^2 increases, the rectangles overshoot, and $R_n > \text{area}(S)$. With left endpoints instead, the rectangles fall short, and $L_n = \frac{(n-1)(2n-1)}{6n^2} < \text{area}(S)$. As $n \rightarrow \infty$, both tend to $\frac{2}{6} = \frac{1}{3}$. The area is squeezed between them, so it must be $\frac{1}{3}$.

n	L_n	R_n	$R_n - L_n = \frac{1}{n}$
10	0.285000	0.385000	0.1
100	0.328350	0.338350	0.01
1000	0.332834	0.333834	0.001

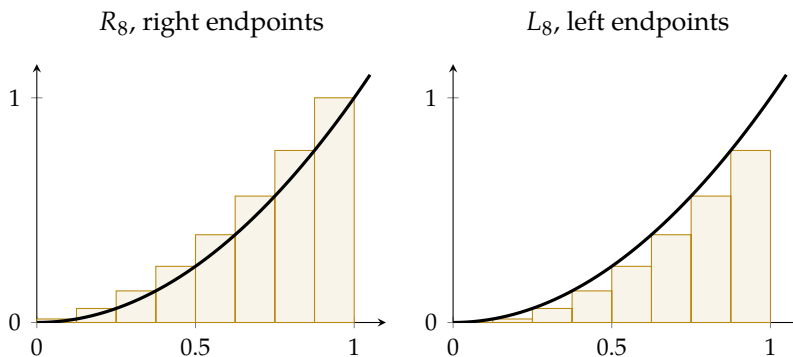


Figure 7.1: Upper and lower rectangle sums for the area under $y = x^2$ on $[0, 1]$. Their difference is the area of the last rectangle on the right, $\frac{1}{n}$.

7.1.2 SIGMA NOTATION AND POWER SUMS

$\sum_{i=m}^n a_i$ denotes $a_m + a_{m+1} + \cdots + a_n$. Sums are linear: $\sum (ca_i + b_i) = c \sum a_i + \sum b_i$.

Theorem 7.1 (Power sums).

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Proof. For the first, write the sum forward and backward and add: each of the n columns sums to $n+1$. For the second, use the telescoping identity $(i+1)^3 -$

$i^3 = 3i^2 + 3i + 1$. Summing from $i = 1$ to n , the left side collapses to $(n + 1)^3 - 1$, so $(n + 1)^3 - 1 = 3 \sum i^2 + 3 \cdot \frac{n(n+1)}{2} + n$, and solving gives the formula. The third follows the same way from $(i + 1)^4 - i^4$, or by induction. ■

7.1.3 THE DISTANCE PROBLEM

If velocity is constant, distance is velocity times time. If velocity varies, divide the time interval into short pieces, on each of which the velocity is nearly constant, and add: distance $\approx \sum v(t_i) \Delta t$. As the pieces shrink the approximation becomes exact. This is the same sum as for area: the distance traveled is the area under the velocity graph.

7.2 THE DEFINITE INTEGRAL

Definition 7.2 (Riemann sum; definite integral). Let f be defined on $[a, b]$. A *partition* P of $[a, b]$ is a set of points $a = x_0 < x_1 < \cdots < x_n = b$; let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max \Delta x_i$. Choose *sample points* $x_i^* \in [x_{i-1}, x_i]$. The *Riemann sum* is $\sum_{i=1}^n f(x_i^*) \Delta x_i$. If there is a number I such that for every $\epsilon > 0$ there is $\delta > 0$ with

$$\left| \sum_{i=1}^n f(x_i^*) \Delta x_i - I \right| < \epsilon$$

for every partition with $\|P\| < \delta$ and every choice of sample points, then f is *integrable* on $[a, b]$, and $I = \int_a^b f(x) dx$ is its *definite integral*.

The integral sign $\int$ is an elongated S, for *summa*, introduced by Leibniz; dx is the descendant of Δx . The variable x is a dummy: $\int_a^b f(x) dx = \int_a^b f(t) dt$. When $f \geq 0$, the integral is the area under the graph. In general it is the *net* area: area above the axis minus area below.

Theorem 7.3 (Integrability). *If f is continuous on $[a, b]$, or bounded with finitely many discontinuities, then f is integrable on $[a, b]$.*

The proof, which uses the uniform continuity of continuous functions on closed intervals, is in Appendix A. Not every bounded function is integrable: the function D of Chapter 2, equal to 1 on rationals and 0 on irrationals, has Riemann sums equal to 1 or 0 depending on the choice of sample points, however fine the partition.

For integrable f we may use any convenient partitions, for instance n equal subintervals with right endpoints:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x, \quad \Delta x = \frac{b-a}{n}, \quad x_i = a + i \Delta x.$$

Example 7.4 (An integral from the definition). Evaluate $\int_0^3 (x^3 - 6x) dx$ from the definition.

Solution. $\Delta x = \frac{3}{n}$, $x_i = \frac{3i}{n}$. The Riemann sum is

$$\sum_{i=1}^n \left(\frac{27i^3}{n^3} - \frac{18i}{n} \right) \frac{3}{n} = \frac{81}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \frac{54}{n^2} \cdot \frac{n(n+1)}{2} \rightarrow \frac{81}{4} - 27 = -\frac{27}{4}.$$

The integral is negative because the region below the axis (on $(0, \sqrt{6})$) has more area than the region above (on $(\sqrt{6}, 3)$).

7.2.1 PROPERTIES OF THE INTEGRAL

Theorem 7.5 (Properties). For integrable f, g and constants c :

- (a) $\int_a^b (cf + g) dx = c \int_a^b f dx + \int_a^b g dx$;
- (b) $\int_a^b f dx + \int_b^c f dx = \int_a^c f dx$, with the conventions $\int_a^a f = 0$ and $\int_b^a f = -\int_a^b f$;
- (c) if $f \leq g$ on $[a, b]$, then $\int_a^b f \leq \int_a^b g$; in particular, if $m \leq f \leq M$, then $m(b-a) \leq \int_a^b f \leq M(b-a)$;
- (d) $|\int_a^b f dx| \leq \int_a^b |f| dx$.

Each follows from the corresponding property of finite sums by taking limits. Property (d) is the triangle inequality for integrals.

Definition 7.6 (Average value). The average value of f on $[a, b]$ is $f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$.

Theorem 7.7 (Mean Value Theorem for integrals). If f is continuous on $[a, b]$, there is $c \in [a, b]$ with $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$.

Proof. By the EVT, f has a minimum m and maximum M on $[a, b]$. By (c), $m \leq f_{\text{avg}} \leq M$. By the IVT, f takes the value f_{avg} somewhere. ■

7.3 THE FUNDAMENTAL THEOREM OF CALCULUS

Theorem 7.8 (Fundamental Theorem of Calculus). *Let f be continuous on $[a, b]$.*

Part 1. *The function $g(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) = f(x)$.*

Part 2. *If F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$.*

Proof. Part 1. For $h \neq 0$, by property (b), $g(x+h) - g(x) = \int_x^{x+h} f(t) dt$. By the Mean Value Theorem for integrals, this equals $f(c_h)h$ for some c_h between x and $x+h$. So $\frac{g(x+h)-g(x)}{h} = f(c_h)$. As $h \rightarrow 0$, $c_h \rightarrow x$, and $f(c_h) \rightarrow f(x)$ by continuity.

Part 2. By Part 1, g is an antiderivative of f . Any other antiderivative is $F = g + C$ (Theorem 5.8). Then $F(b) - F(a) = g(b) - g(a) = \int_a^b f dt - 0$. ■

Part 1 says that every continuous function has an antiderivative, namely its accumulation function. Part 2 says that to compute an integral we need only find one antiderivative. Together: differentiation and integration undo each other. We write $F(x) \Big|_a^b$ or $[F(x)]_a^b$ for $F(b) - F(a)$.

Example 7.9 (Evaluating integrals). (a) $\int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$, the area of Fig. 7.1, in one line. (b) $\int_0^3 (x^3 - 6x) dx = \left[\frac{x^4}{4} - 3x^2 \right]_0^3 = \frac{81}{4} - 27 = -\frac{27}{4}$, as in Example 7.4. (c) $\int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = 2$. (d) $\int_1^e \frac{dx}{x} = \ln x \Big|_1^e = 1$. (e) $\int_0^1 \frac{dx}{1+x^2} = \arctan 1 - \arctan 0 = \frac{\pi}{4}$.

COMMON ERROR. FTC Part 2 requires f continuous on the whole interval. $\int_{-1}^1 \frac{dx}{x^2}$ is not $\left[-\frac{1}{x} \right]_{-1}^1 = -2$: the integrand is positive, and the “answer” is negative. The function $\frac{1}{x^2}$ is unbounded near 0, not integrable, and the integral diverges (Chapter 9).

Example 7.10 (Differentiating an accumulation function). Find $\frac{d}{dx} \int_1^{x^3} \sqrt{1+t^4} dt$ and $\frac{d}{dx} \int_{\sin x}^{x^2} e^{-t^2} dt$.

Solution. By Part 1 and the chain rule, with $u = x^3$: $\sqrt{1+u^4} \frac{du}{dx} = 3x^2 \sqrt{1+x^{12}}$. For the second, split at 0: $\int_0^{x^2} - \int_0^{\sin x}$, so the derivative is $e^{-x^4} \cdot 2x - e^{-\sin^2 x} \cos x$. In general $\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x))b'(x) - f(a(x))a'(x)$. No antiderivative of e^{-t^2} in elementary terms exists, and none was needed.

Part 1 also defines new functions. The *error function* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ of probability, and the *Fresnel integral* $S(x) = \int_0^x \sin \frac{\pi t^2}{2} dt$ of optics, are not elementary, but they are as well-defined, as computable, and as differentiable as $\sin x$.

7.3.1 NET CHANGE

Part 2 says that the integral of a rate of change is the net change: $\int_a^b F'(x) dx = F(b) - F(a)$. If $v(t)$ is velocity, $\int_{t_1}^{t_2} v dt$ is the displacement, and $\int_{t_1}^{t_2} |v| dt$ the total distance. If $r(t)$ is the rate at which water flows into a tank, $\int_{t_1}^{t_2} r dt$ is the amount that flows in.

Example 7.11 (Displacement and distance). For $v(t) = t^2 - t - 6$ m/s on $[1, 4]$: the displacement is $\int_1^4 v dt = [\frac{t^3}{3} - \frac{t^2}{2} - 6t]_1^4 = -\frac{9}{2}$ m. Since $v = (t - 3)(t + 2)$ is negative on $[1, 3)$ and positive on $(3, 4]$, the distance is $-\int_1^3 v dt + \int_3^4 v dt = \frac{22}{3} + \frac{17}{6} = \frac{61}{6} \approx 10.17$ m.

7.4 THE SUBSTITUTION RULE

Every differentiation rule, read backward, is an integration rule. The chain rule $\frac{d}{dx} F(g(x)) = F'(g(x))g'(x)$ becomes:

Theorem 7.12 (Substitution rule). If $u = g(x)$ is differentiable and f is continuous on the range of g , then

$$\int f(g(x)) g'(x) dx = \int f(u) du, \quad \int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Proof. If $F' = f$, then $\frac{d}{dx} F(g(x)) = f(g(x))g'(x)$ by the chain rule, so both sides of the first formula equal $F(g(x)) + C$. For the second, both sides equal $F(g(b)) -$

$F(g(a))$ by the Fundamental Theorem. ■

In practice: choose u as an inner function whose derivative (up to a constant) appears as a factor; write $du = g'(x) dx$; rewrite everything in terms of u ; integrate; substitute back. The differential notation makes the bookkeeping automatic.

Example 7.13 (Substitutions). Evaluate (a) $\int x^2 \sqrt{x^3 + 1} dx$, (b) $\int \tan x dx$, (c) $\int \frac{\ln x}{x} dx$, (d) $\int \frac{dx}{\sqrt{9-x^2}}$, (e) $\int_0^2 \frac{x dx}{(1+x^2)^2}$.

Solution. (a) $u = x^3 + 1$, $du = 3x^2 dx$: $\frac{1}{3} \int \sqrt{u} du = \frac{2}{9} (x^3 + 1)^{3/2} + C$.

(b) $\tan x = \frac{\sin x}{\cos x}$; $u = \cos x$, $du = -\sin x dx$: $-\int \frac{du}{u} = -\ln |\cos x| + C = \ln |\sec x| + C$.

(c) $u = \ln x$, $du = \frac{dx}{x}$: $\int u du = \frac{1}{2} \ln^2 x + C$.

(d) $x = 3u$ (a substitution in the reverse direction, $u = x/3$): $\int \frac{3 du}{3\sqrt{1-u^2}} = \arcsin \frac{x}{3} + C$.

(e) $u = 1 + x^2$, $du = 2x dx$, and the limits change: $x = 0 \mapsto u = 1$, $x = 2 \mapsto u = 5$. So $\frac{1}{2} \int_1^5 u^{-2} du = \frac{1}{2} \left[-\frac{1}{u} \right]_1^5 = \frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$.

COMMON ERROR. In a definite integral, change the limits along with the variable, or return to x before evaluating, but never mix the two: evaluating an antiderivative in u at the x -limits is the most common substitution error.

Theorem 7.14 (Symmetry). If f is continuous on $[-a, a]$: if f is even, $\int_{-a}^a f = 2 \int_0^a f$; if f is odd, $\int_{-a}^a f = 0$.

Proof. Split at 0 and substitute $u = -x$ in $\int_{-a}^0 f(x) dx = \int_0^a f(-u) du$, which is $\int_0^a f$ or $-\int_0^a f$. ■

So $\int_{-2}^2 x^5 \cos x dx = 0$ without computation, since the integrand is odd.

7.5 THE LOGARITHM DEFINED AS AN INTEGRAL

Until now we have used e^x and $\ln x$ on the strength of facts about exponentials we did not prove: that b^x makes sense for irrational x , that $\frac{b^h - 1}{h}$ has a limit, that this limit depends continuously on b . The integral lets us build both functions from scratch, and prove every one of those facts.

Definition 7.15 (Natural logarithm). For $x > 0$, $\ln x = \int_1^x \frac{dt}{t}$.

Theorem 7.16 (Properties of the logarithm). (a) $\ln$ is differentiable with $(\ln x)' = \frac{1}{x}$; it is increasing and concave down, and $\ln 1 = 0$.

(b) $\ln(xy) = \ln x + \ln y$, $\ln \frac{x}{y} = \ln x - \ln y$, and $\ln(x^r) = r \ln x$ for rational r .

(c) $\lim_{x \rightarrow \infty} \ln x = \infty$ and $\lim_{x \rightarrow 0^+} \ln x = -\infty$; the range of $\ln$ is $\mathbb{R}$.

Proof. (a) By the FTC, $(\ln x)' = \frac{1}{x} > 0$, and $(\ln x)'' = -\frac{1}{x^2} < 0$.

(b) Fix $y > 0$. The functions $\ln(xy)$ and $\ln x$ have the same derivative, $\frac{y}{xy} = \frac{1}{x}$, so they differ by a constant: $\ln(xy) = \ln x + C$. At $x = 1$, $C = \ln y$. Similarly $\ln(x^r)$ and $r \ln x$ both have derivative $\frac{r}{x}$ and vanish at $x = 1$. The quotient rule follows by applying the product rule to $\frac{x}{y} \cdot y = x$.

(c) $\ln 2 > 0$, and $\ln(2^n) = n \ln 2 \rightarrow \infty$; since $\ln$ is increasing, $\ln x \rightarrow \infty$. Then $\ln \frac{1}{x} = -\ln x \rightarrow -\infty$ as $x \rightarrow 0^+$. By continuity and the IVT, $\ln$ takes every real value. ■

Definition 7.17 (e and the exponential). Since $\ln : (0, \infty) \rightarrow \mathbb{R}$ is increasing and onto, it has an inverse, $\exp : \mathbb{R} \rightarrow (0, \infty)$. Define $e = \exp(1)$, the number with $\ln e = 1$, and for any real x and $b > 0$, define $b^x = \exp(x \ln b)$.

For rational $x = r$, $\exp(r) = e^r$ in the old sense, since $\ln(e^r) = r \ln e = r$. So $\exp(x) = e^x$ extends powers of e to all real exponents, and $b^x = e^{x \ln b}$ is the definition of irrational powers. Now everything follows rigorously:

- $\frac{d}{dx} e^x = e^x$, by Theorem 4.8: $(\exp)'(x) = \frac{1}{\ln'(e^x)} = e^x$.
- The laws of exponents, from the laws of logarithms.
- $\frac{d}{dx} b^x = b^x \ln b$, so $\lim_{h \rightarrow 0} \frac{b^h - 1}{h} = \ln b$, continuous in b and equal to 1 exactly when $b = e$, as claimed in Chapter 3.
- $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$, by Theorem 4.14, whose proof used only $\ln'(1) = 1$.

Geometrically, $\ln x$ is the area under the hyperbola $y = \frac{1}{t}$ from 1 to x , and e is the point where that area reaches 1. In the seventeenth century the logarithm was in fact first understood as a hyperbolic area, by Grégoire de Saint-Vincent and Alphonse de Sarasa.

7.6 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 7.1 A limit of sums as an integral

Evaluate $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n}{n^2 + i^2}$ and $\lim_{n \rightarrow \infty} \frac{1^4 + 2^4 + \cdots + n^4}{n^5}$.

Solution. Rewrite the first as a Riemann sum: $\frac{n}{n^2 + i^2} = \frac{1}{n} \cdot \frac{1}{1 + (i/n)^2}$. With $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$, this is the right-endpoint sum for $\int_0^1 \frac{dx}{1+x^2}$. Since $\frac{1}{1+x^2}$ is continuous, the limit is the integral, $\arctan 1 = \frac{\pi}{4}$. A formula for π from a limit of rational numbers.

The second is $\sum \left(\frac{i}{n}\right)^4 \frac{1}{n} \rightarrow \int_0^1 x^4 dx = \frac{1}{5}$. The method converts limits of sums, which are hard, into integrals, which the FTC makes easy.

— Problem 7.2 Average value and the MVT for integrals

Find the average value of $f(x) = \sin^2 x$ on $[0, \pi]$, and every c at which $f(c)$ equals it.

Solution. Using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\int_0^\pi \sin^2 x dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi = \frac{\pi}{2}$, so the average is $\frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$. Then $\sin^2 c = \frac{1}{2}$ gives $\sin c = \pm \frac{1}{\sqrt{2}}$, and on $[0, \pi]$: $c = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. The average of $\sin^2$ over any whole number of half-periods is $\frac{1}{2}$, which is why alternating-current power is computed with the root-mean-square voltage $V_{\max}/\sqrt{2}$.

— Problem 7.3 An accumulation function

Let $g(x) = \int_0^x (t^2 - 4)e^{-t} dt$. Find the intervals on which g increases, its local extrema, and the intervals of concavity.

Solution. By the FTC, $g'(x) = (x^2 - 4)e^{-x}$, which is negative on $(-2, 2)$ and positive outside. So g increases on $(-\infty, -2)$ and $(2, \infty)$ and decreases on $(-2, 2)$: a local maximum at -2 and a local minimum at 2 . Then $g''(x) = (2x - x^2 + 4)e^{-x}$, with zeros at $x = 1 \pm \sqrt{5}$: concave down for $x < 1 - \sqrt{5}$ and $x > 1 + \sqrt{5}$, up between. All of this without evaluating the integral, which is possible here (by the parts formula of Chapter 9) but not needed.

— **Problem 7.4** A substitution that must be found

Evaluate $\int \frac{dx}{x \ln x \ln(\ln x)}$ and $\int_0^{\pi/4} \frac{\sec^2 x}{1 + \tan x} dx$.

Solution. For the first, $u = \ln(\ln x)$ gives $du = \frac{1}{\ln x} \cdot \frac{1}{x} dx$, exactly the remaining factor. So the integral is $\int \frac{du}{u} = \ln |\ln(\ln x)| + C$. For the second, $u = 1 + \tan x$, $du = \sec^2 x dx$, limits 1 to 2: $\int_1^2 \frac{du}{u} = \ln 2$. In both cases the derivative of the substituted expression is a factor of the integrand. That observation, not trial and error, is how substitutions are found.

— **Problem 7.5** Bounding an integral

Show that $\frac{1}{e} \leq \int_0^1 e^{-x^2} dx \leq 1$, and then sharpen the lower bound to $\frac{2}{3}$.

Solution. On $[0, 1]$, $e^{-1} \leq e^{-x^2} \leq 1$, so by property (c) the integral lies between $\frac{1}{e} \approx 0.368$ and 1. For a sharper lower bound, use $e^u \geq 1 + u$ (the worked problems of Chapter 5) with $u = -x^2$: $e^{-x^2} \geq 1 - x^2$, so $\int_0^1 e^{-x^2} dx \geq \int_0^1 (1 - x^2) dx = \frac{2}{3} \approx 0.667$. The true value is 0.746824. The integrand has no elementary antiderivative, and comparison with simpler functions is the main way to bound it without numerical methods.

— **Problem 7.6** The derivative of an integral with variable limits

Find $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \sin(t^2) dt$.

Solution. The form is $\frac{0}{0}$, since the integral of a bounded function over an interval of length x tends to 0. By L'Hôpital's rule and the FTC, $\lim \frac{\sin(x^2)}{3x^2} = \frac{1}{3}$. Intuitively, $\sin(t^2) \approx t^2$ near 0, so the integral is about $\frac{x^3}{3}$.

— **Problem 7.7** The logarithm from the integral, numerically

Show from Definition 7.15 alone that $\frac{1}{2} < \ln 2 < 1$ and $\ln 4 > 1$, and conclude that $2 < e < 4$.

Solution. On $[1, 2]$, $\frac{1}{2} \leq \frac{1}{t} \leq 1$, with strict inequality except at the endpoints, so $\frac{1}{2} < \int_1^2 \frac{dt}{t} < 1$. Then $\ln 4 = 2 \ln 2 > 1 = \ln e$. Since $\ln$ is increasing, $\ln 2 < 1 < \ln 4$ gives $2 < e < 4$. Finer partitions sharpen this: the trapezoid estimate with four strips gives $\ln 2 \approx 0.6970$ (true value 0.6931). Computing logarithms as areas was the method of the first logarithm tables after Napier and Briggs.

EXERCISES

A. COMPUTATION

7.1 Compute R_4 and L_4 for $f(x) = \frac{1}{x}$ on $[1, 3]$, and explain which is an overestimate.

7.2 Find $\int_0^2 (x^2 + 1) dx$ as a limit of right-endpoint Riemann sums.

7.3 Express $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$ as an integral and evaluate it.

7.4 Use properties of integrals to show $2 \leq \int_{-1}^1 \sqrt{1+x^2} dx \leq 2\sqrt{2}$.

7.5 Evaluate $\int_1^4 \left(\sqrt{x} - \frac{2}{x^2} \right) dx$.

7.6 Evaluate $\int_0^{\pi/3} \sec^2 \theta d\theta$ and $\int_0^{\pi/4} \sec x \tan x dx$.

7.7 Evaluate $\int_{-1}^2 |x^2 - 1| dx$.

7.8 Evaluate $\int_0^1 \frac{x^2 + 2}{x^2 + 1} dx$.

7.9 Find $g'(x)$ for $g(x) = \int_2^x \cos(t^3) dt$ and $h(x) = \int_x^{x^2} \frac{dt}{\ln t}$ ($x > 1$).

7.10 Find the average value of $f(x) = \frac{1}{x}$ on $[1, e^2]$.

7.11 A particle has velocity $v(t) = 3t^2 - 12t + 9$ m/s. Find the displacement and the distance traveled on $[0, 4]$.

7.12 Water flows into a tank at $r(t) = 200 - 4t$ L/min for $0 \leq t \leq 50$. How much enters in the first 50 minutes?

7.13 Evaluate $\int x(2x^2 + 3)^4 dx$.

7.14 Evaluate $\int \frac{\cos x}{\sqrt{1 + \sin x}} dx$.

7.15 Evaluate $\int e^x \sqrt{1 + e^x} dx$.

7.16 Evaluate $\int \frac{dx}{x(\ln x)^2}$.

7.17 Evaluate $\int \frac{x dx}{x^4 + 1}$.

7.18 Evaluate $\int \cot x dx$ and $\int \frac{\sin(\ln x)}{x} dx$.

7.19 Evaluate $\int_0^1 x e^{-x^2} dx$.

7.20 Evaluate $\int_0^{\pi/2} \cos x \sin^5 x dx$.

7.21 Evaluate $\int_1^e \frac{(\ln x)^2}{x} dx$.

7.22 Evaluate $\int_0^4 \frac{dx}{\sqrt{2x+1}}$.

7.23 Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$ and $\int_0^{\sqrt{3}} \frac{dx}{9+x^2}$.

7.24 Evaluate $\int_{-\pi}^{\pi} \frac{x^3}{1+x^2} \cos x dx$ without computation.

7.25 Evaluate $\int_0^1 x \sqrt{1-x} dx$ with $u = 1-x$.

7.26 Evaluate $\int_1^2 \frac{dx}{x^2} e^{1/x}$.

7.27 Find f if $f(x) = 1 + \int_0^x f(t) dt$ for all x . (Differentiate.)

7.28 If $\int_0^x f(t) dt = x \sin \pi x$, find $f(2)$.

7.29 Find the interval on which $F(x) = \int_0^x \frac{t^2 - 1}{t^2 + 1} dt$ is decreasing.

7.30 Use the definition $\ln x = \int_1^x \frac{dt}{t}$ and a Riemann sum with 4 strips (mid-points) to estimate $\ln 3$.

B. THEORY AND PROOF

7.31 Prove $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ by induction.

- 7.32** Prove that if f is continuous, $f \geq 0$ on $[a, b]$, and $\int_a^b f = 0$, then $f \equiv 0$ on $[a, b]$.
- 7.33** Prove that if f is periodic with period p and continuous, then $\int_a^{a+p} f$ is independent of a .
- 7.34** Prove that $\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx$.
- 7.35** Prove that $\ln(x^r) = r \ln x$ for all real r , using the definition $x^r = e^{r \ln x}$.
- 7.36** Show that $\int_a^{ab} \frac{dt}{t} = \int_1^b \frac{dt}{t}$ by substitution, and deduce $\ln(ab) = \ln a + \ln b$ geometrically.
- 7.37** Prove that if f is continuous and $\int_a^b f(x)g(x) dx = 0$ for every continuous g , then $f \equiv 0$.
- 7.38** Prove that the Dirichlet function is not Riemann integrable on $[0, 1]$.

C. CHALLENGE

- 7.39** Evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln \left(1 + \frac{i}{n} \right)$, and deduce that

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{(n+1)(n+2) \cdots (2n)}}{n} = \frac{4}{e}.$$

- 7.40** Prove that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$ by the substitution $u = \frac{\pi}{2} - x$.
- 7.41** Evaluate $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$. (Substitute $x = \tan \theta$, then $\theta \mapsto \frac{\pi}{4} - \theta$; the answer is $\frac{\pi}{8} \ln 2$.)
- 7.42** Prove that $\frac{1}{n+1} < \ln \frac{n+1}{n} < \frac{1}{n}$ using the integral definition, and deduce that $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$ is decreasing and positive.
- 7.43** Show that $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi$, and deduce that $\pi < \frac{22}{7}$.

CHAPTER 8

APPLICATIONS OF INTEGRATION

LEARNING OBJECTIVES

- Compute areas between curves, integrating in x or in y .
- Find volumes of solids by slicing, by disks and washers, and by cylindrical shells.
- Derive and apply the formulas for arc length and for the area of a surface of revolution.
- Set up integrals for work, hydrostatic force, moments, and centers of mass, and apply the theorem of Pappus.
- Use probability density functions to compute probabilities, means, and medians.

Every application in this chapter follows one pattern. A quantity Q is built up from contributions spread over an interval. Cut the interval into small pieces; on each piece approximate the contribution, $\Delta Q \approx f(x_i^*) \Delta x$, by treating something that varies as constant; add the pieces to get a Riemann sum; and let the pieces shrink, so that the sum becomes the integral $Q = \int_a^b f(x) dx$. In physics and engineering this is done in one breath, by writing $dQ = f(x) dx$ for the contribution of an “infinitesimal” piece and then integrating. The notation is shorthand for the limit argument, and the art lies in finding the right f .

8.1 AREAS BETWEEN CURVES

If $f(x) \geq g(x)$ on $[a, b]$, the region between their graphs is approximated by vertical strips of height $f(x_i^*) - g(x_i^*)$ and width Δx , so its area is

$$A = \int_a^b (f(x) - g(x)) dx.$$

If the curves cross, the integrand must be $|f - g|$: split the interval at the crossings. When the region is more naturally bounded by curves $x = f(y)$ and $x = g(y)$ on the left and right, integrate in y with horizontal strips: $A = \int_c^d (f(y) - g(y)) dy$.

Example 8.1 (Two parabolas). Find the area enclosed by $y = x^2$ and $y = 2x - x^2$.

Solution. They meet where $x^2 = 2x - x^2$, at $x = 0$ and $x = 1$. On $[0, 1]$ the second is on top: $A = \int_0^1 (2x - 2x^2) dx = [x^2 - \frac{2}{3}x^3]_0^1 = \frac{1}{3}$.

Example 8.2 (Integrating with respect to y). Find the area enclosed by $y^2 = 2x + 6$ and $y = x - 1$.

Solution. In x , the lower boundary changes formula at $x = -1$, and two integrals would be needed. In y : the parabola is $x = \frac{y^2}{2} - 3$ (left) and the line is $x = y + 1$ (right). They meet where $y + 1 = \frac{y^2}{2} - 3$, that is, $y^2 - 2y - 8 = 0$, at $y = -2$ and $y = 4$. So $A = \int_{-2}^4 (y + 1 - \frac{y^2}{2} + 3) dy = [\frac{y^2}{2} + 4y - \frac{y^3}{6}]_{-2}^4 = \frac{40}{3} - (-\frac{14}{3}) = 18$.

8.2 VOLUMES

8.2.1 VOLUMES BY SLICING

If a solid lies between $x = a$ and $x = b$, and its cross-section by the plane perpendicular to the x -axis at x has area $A(x)$, then a slab of thickness Δx has volume about $A(x)\Delta x$, and

$$V = \int_a^b A(x) dx.$$

This is *Cavalieri's principle* in quantitative form: solids with equal cross-sectional areas at every height have equal volumes.

Example 8.3 (A pyramid and a sphere). (a) A pyramid with square base of side s and height h : at distance y below the apex the cross-section is a square of side $\frac{sy}{h}$, so $V = \int_0^h \frac{s^2 y^2}{h^2} dy = \frac{1}{3}s^2 h$. (b) A ball of radius r : the cross-section at x is a disk of radius $\sqrt{r^2 - x^2}$, so $V = \int_{-r}^r \pi(r^2 - x^2) dx = \frac{4}{3}\pi r^3$.

Both formulas were known to the Greeks; Archimedes regarded the sphere as his greatest achievement.

8.2.2 DISKS AND WASHERS

When the solid is obtained by revolving a region about an axis, the cross-sections perpendicular to the axis are disks, or washers if the region does not touch the axis. Revolving the region under $y = f(x) \geq 0$ about the x -axis gives $A(x) = \pi f(x)^2$. Revolving the region between $f \geq g \geq 0$ gives washers with outer radius f and inner radius g :

$$V = \int_a^b \pi(f(x)^2 - g(x)^2) dx.$$

Example 8.4 (Washers about a horizontal line). The region between $y = x$ and $y = x^2$ is revolved about the line $y = 2$. Find the volume.

Solution. The curves meet at $x = 0, 1$, with $x \geq x^2$ on $[0, 1]$. Distances are measured from the axis $y = 2$: the outer radius is $2 - x^2$ (the curve farther from the axis) and the inner radius $2 - x$. So

$$\begin{aligned} V &= \pi \int_0^1 [(2 - x^2)^2 - (2 - x)^2] dx = \pi \int_0^1 (x^4 - 5x^2 + 4x) dx \\ &= \pi \left(\frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8\pi}{15}. \end{aligned}$$

8.2.3 CYLINDRICAL SHELLS

Revolving a thin vertical strip at distance x from the axis, of height $h(x)$ and width Δx , produces a thin cylindrical shell. Cut and unrolled, it is nearly a flat slab of dimensions $2\pi x \times h(x) \times \Delta x$. So revolving the region under $y = f(x)$, $0 \leq a \leq x \leq b$, about the y -axis gives

$$V = \int_a^b 2\pi x f(x) dx.$$

Example 8.5 (Shells versus washers). Find the volume obtained by revolving the region under $y = 2x^2 - x^3$, $0 \leq x \leq 2$, about the y -axis.

Solution. Washers would require solving $y = 2x^2 - x^3$ for x , a cubic. Shells

avoid it:

$$V = \int_0^2 2\pi x(2x^2 - x^3) dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5} \right]_0^2 = 2\pi \left(8 - \frac{32}{5} \right) = \frac{16\pi}{5}.$$

The choice between methods is the choice between strips parallel and perpendicular to the axis. Use whichever makes the integrand easier, or the boundary a single formula.

Example 8.6 (A torus). The disk $(x - R)^2 + y^2 \leq r^2$, with $R > r$, is revolved about the y -axis. Find the volume of the resulting torus.

Solution. By shells, the strip at x has height $2\sqrt{r^2 - (x - R)^2}$, so $V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x - R)^2} dx$. Substitute $u = x - R$: $V = 4\pi \int_{-r}^r (u + R)\sqrt{r^2 - u^2} du$. The term with u is odd and integrates to 0. The other is $4\pi R$ times the area of a half-disk, $\frac{1}{2}\pi r^2$. So $V = 2\pi^2 R r^2 = (2\pi R)(\pi r^2)$: the area of the disk times the distance its center travels. This is the theorem of Pappus (Theorem 8.16).

8.3 ARC LENGTH

Approximate a curve $y = f(x)$, $a \leq x \leq b$, by a polygon with vertices on the curve. The segment over $[x_{i-1}, x_i]$ has length $\sqrt{\Delta x^2 + \Delta y_i^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x$, and by the Mean Value Theorem $\frac{\Delta y_i}{\Delta x} = f'(x_i^*)$ for some x_i^* in the subinterval. So the polygon's length is a Riemann sum for:

Theorem 8.7 (Arc length). If f' is continuous on $[a, b]$, the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

The differential form $ds = \sqrt{dx^2 + dy^2}$ is the Pythagorean theorem for an infinitesimal triangle. Arc length integrals are rarely elementary: the square root usually prevents it. The examples that work are chosen so that $1 + f'^2$ is a perfect square.

Example 8.8 (A semicubical parabola and a catenary). Find the length of (a) $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$; (b) the catenary $y = \cosh x$ from $x = -1$ to $x = 1$.

Solution. (a) $1 + y'^2 = 1 + \frac{9}{4}x$, so $L = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx = \frac{4}{9} \cdot \frac{2}{3} (1 + \frac{9}{4}x)^{3/2} \Big|_0^4 = \frac{8}{27} (10\sqrt{10} - 1) \approx 9.073$. The chord has length $\sqrt{80} \approx 8.944$; the curve is only slightly longer.

(b) $1 + \sinh^2 x = \cosh^2 x$, so $L = \int_{-1}^1 \cosh x dx = 2 \sinh 1 \approx 2.350$.

COMMON ERROR. $\int \sqrt{1 + f'^2} dx$ is not $\int (1 + f') dx$: square roots do not distribute over sums. Most arc length integrals must be computed numerically (Chapter 9). The circumference of an ellipse, for instance, is an *elliptic integral*, which has no elementary formula.

8.4 AREA OF A SURFACE OF REVOLUTION

Revolving a short segment of length Δs at distance r from the axis sweeps out a band of a cone, a *frustum*, whose area is approximately $2\pi r \Delta s$ (circumference times slant width). Hence:

$$S = \int 2\pi r ds, \quad \text{about the } x\text{-axis: } S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx.$$

Example 8.9 (Archimedes' hat-box theorem). Find the area of the zone of a sphere of radius R between two parallel planes a distance h apart.

Solution. Revolve $y = \sqrt{R^2 - x^2}$ about the x -axis. Then $y' = -\frac{x}{y}$ and $y\sqrt{1 + y'^2} = \sqrt{y^2 + x^2} = R$. So the area between $x = c$ and $x = c + h$ is $\int_c^{c+h} 2\pi R dx = 2\pi Rh$, independent of c . Taking $h = 2R$ gives $4\pi R^2$. The zone has the same area as the band of the circumscribed cylinder between the same planes, a result Archimedes had engraved on his tomb.

Example 8.10 (Gabriel's horn). Revolve $y = \frac{1}{x}$, $x \geq 1$, about the x -axis. Show that the solid has finite volume but its surface has infinite area.

Solution. $V = \int_1^\infty \pi \frac{1}{x^2} dx = \pi \lim_{b \rightarrow \infty} (1 - \frac{1}{b}) = \pi$. The area is $S = \int_1^\infty \frac{2\pi}{x} \sqrt{1 + \frac{1}{x^4}} dx \geq \int_1^\infty \frac{2\pi}{x} dx = \infty$. A horn that can be filled with π units

of paint, but whose inside surface cannot be painted. (Integrals over infinite intervals are made precise in Chapter 9.) The paradox dissolves on reflection: a finite volume of paint spread to a layer of any fixed thickness covers only finite area, and the horn's interior narrows below every thickness.

8.5 WORK

A constant force F moving an object a distance d along its line of action does work $W = Fd$. In SI units, force is in newtons and work in joules (N·m). If the force varies with position, $F = F(x)$, the work over a short displacement Δx is about $F(x)\Delta x$, and

$$W = \int_a^b F(x) dx.$$

Example 8.11 (Stretching a spring). A force of 40 N holds a spring stretched 0.1 m beyond its natural length. How much work stretches it from 0.1 m to 0.25 m beyond natural length?

Solution. By Hooke's law, $F(x) = kx$, where x is the stretch. Then $40 = 0.1k$, $k = 400$ N/m, and $W = \int_{0.1}^{0.25} 400x dx = 200(0.0625 - 0.01) = 10.5$ J. In general the energy stored in a spring stretched by x is $\frac{1}{2}kx^2$.

When different parts of an object move different distances, slice the *object*, not the path.

Example 8.12 (Pumping out a tank). A tank shaped as an inverted cone, 10 m tall with top radius 4 m, is full of water. How much work pumps all the water out over the top? (Water weighs $\rho g = 9800$ N/m³.)

Solution. Measure y upward from the vertex. The horizontal slice at height y has radius $\frac{4y}{10} = 0.4y$ and thickness Δy , so its weight is $9800\pi(0.4y)^2\Delta y$, and it must be lifted a distance $10 - y$. So

$$\begin{aligned} W &= \int_0^{10} 9800\pi(0.16y^2)(10 - y) dy = 1568\pi \left[\frac{10y^3}{3} - \frac{y^4}{4} \right]_0^{10} \\ &= 1568\pi \cdot \frac{2500}{3} \approx 4.10 \times 10^6 \text{ J.} \end{aligned}$$

Example 8.13 (Escaping the Earth). The gravitational force on a mass m at distance r from the Earth's center is $F = \frac{GMm}{r^2}$. The work to lift it from the surface $r = R$ to infinity is $\int_R^\infty \frac{GMm}{r^2} dr = \frac{GMm}{R}$, finite although the distance is infinite. Setting this equal to the kinetic energy $\frac{1}{2}mv^2$ gives the escape velocity $v = \sqrt{2GM/R} \approx 11.2$ km/s.

8.6 HYDROSTATIC PRESSURE AND FORCE

At depth y in a fluid of density ρ , the pressure is $P = \rho gy$, the same in all directions (Pascal's principle). On a horizontal plate the force is pressure times area, but on a vertical wall the pressure varies with depth. A horizontal strip at depth y with width $w(y)$ and height Δy feels a force of about $\rho g y w(y) \Delta y$, so

$$F = \int \rho g y w(y) dy.$$

Example 8.14 (A dam). A dam has the shape of a trapezoid 50 m wide at the top, 30 m wide at the bottom, and 20 m tall. Find the force on it when the water reaches the top.

Solution. At depth y below the surface the width is $50 - y$ (it decreases linearly from 50 to 30 over 20 m). So

$$\begin{aligned} F &= \int_0^{20} 9800 y (50 - y) dy = 9800 \left[25y^2 - \frac{y^3}{3} \right]_0^{20} \\ &= 9800 \left(10\,000 - \frac{8000}{3} \right) \approx 7.19 \times 10^7 \text{ N.} \end{aligned}$$

8.7 MOMENTS AND CENTERS OF MASS

Masses m_i at positions x_i on a rod balance at the *center of mass* $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$, where the *moment* $\sum m_i x_i$ is balanced. For a thin plate (*lamina*) of uniform density occupying the region between $f \geq g$ on $[a, b]$, slicing into vertical strips (each with its mass concentrated at the midpoint of its height) gives the *centroid*:

$$\bar{x} = \frac{1}{A} \int_a^b x(f(x) - g(x)) dx, \quad \bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} (f(x)^2 - g(x)^2) dx,$$

where A is the area. The centroid depends only on the shape, not the density, when the density is uniform.

Example 8.15 (A quarter disk). Find the centroid of the quarter disk $x^2 + y^2 \leq r^2$, $x, y \geq 0$.

Solution. By symmetry $\bar{x} = \bar{y}$. With $f(x) = \sqrt{r^2 - x^2}$ and $A = \frac{\pi r^2}{4}$: $\bar{y} = \frac{4}{\pi r^2} \int_0^r \frac{1}{2}(r^2 - x^2) dx = \frac{4}{\pi r^2} \cdot \frac{r^3}{3} = \frac{4r}{3\pi} \approx 0.424r$.

Theorem 8.16 (Pappus). *If a plane region of area A lies on one side of a line in its plane and is revolved about that line, the volume of the resulting solid is $V = 2\pi\bar{r}A$, where $\bar{r}$ is the distance from the centroid to the line.*

Proof. For the y -axis, shells give $V = \int 2\pi x(f - g) dx = 2\pi \int x(f - g) dx = 2\pi\bar{x}A$, by the definition of $\bar{x}$. ■

For the torus, the centroid of the disk travels $2\pi R$, so $V = 2\pi R \cdot \pi r^2$, as found in Example 8.6. Pappus stated the theorem in the fourth century, thirteen hundred years before calculus.

8.8 PROBABILITY

A continuous random quantity X , such as the lifetime of a component or the height of a person, is described by a *probability density function* $f \geq 0$ with $\int_{-\infty}^{\infty} f = 1$: the probability that X lies in $[a, b]$ is $\int_a^b f(x) dx$. Its *mean* is $\mu = \int xf(x) dx$, the center of mass of the density, and its *median* m satisfies $\int_{-\infty}^m f = \frac{1}{2}$.

Example 8.17 (Waiting times). The time T in minutes until a customer arrives has density $f(t) = \frac{1}{5}e^{-t/5}$ for $t \geq 0$. Find the probability that the wait exceeds 10 minutes, the mean wait, and the median wait.

Solution. $P(T > 10) = \int_{10}^{\infty} \frac{1}{5}e^{-t/5} dt = e^{-2} \approx 0.135$. The mean is $\int_0^{\infty} \frac{t}{5}e^{-t/5} dt = 5$ minutes (by integration by parts, Chapter 9). The median solves $1 - e^{-m/5} = \frac{1}{2}$, so $m = 5 \ln 2 \approx 3.47$ minutes. The median is less than the mean: occasional long waits pull the mean up. This exponential distribution is the continuous version of radioactive decay, and its median is the half-life.

The most important density is the *normal* distribution $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$. That $\int f = 1$ requires $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, a fact proved in Volume IV with a double integral. The bell curve of Chapter 5 is this density.

8.9 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 8.1 Area of a region with a crossing

Find the total area between $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \pi$.

Solution. They cross at $x = \frac{\pi}{4}$. On $[0, \frac{\pi}{4}]$, $\cos x \geq \sin x$; on $[\frac{\pi}{4}, \pi]$, $\sin x \geq \cos x$. So

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}. \end{aligned}$$

Integrating $\sin x - \cos x$ over $[0, \pi]$ without splitting gives 2, the net signed area, which is not the area.

— Problem 8.2 A solid with known cross-sections

The base of a solid is the disk $x^2 + y^2 \leq 4$. Cross-sections perpendicular to the x -axis are equilateral triangles. Find the volume.

Solution. At x , the base of the triangle is the chord of length $2\sqrt{4 - x^2}$, and an equilateral triangle of side s has area $\frac{\sqrt{3}}{4}s^2$. So $A(x) = \frac{\sqrt{3}}{4} \cdot 4(4 - x^2) = \sqrt{3}(4 - x^2)$, and

$$V = \int_{-2}^2 \sqrt{3}(4 - x^2) dx = \sqrt{3} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32\sqrt{3}}{3} \approx 18.48.$$

No revolution is involved. The slicing method needs only the area of each cross-section.

— Problem 8.3 Shells about a vertical line

The region bounded by $y = x - x^2$ and $y = 0$ is revolved about the line $x = 2$. Find the volume, by shells and by Pappus.

Solution. *Shells.* A vertical strip at $x \in [0, 1]$ is at distance $2 - x$ from the axis and has height $x - x^2$:

$$\begin{aligned} V &= \int_0^1 2\pi(2 - x)(x - x^2) dx = 2\pi \int_0^1 (2x - 3x^2 + x^3) dx \\ &= 2\pi \left(1 - 1 + \frac{1}{4} \right) = \frac{\pi}{2}. \end{aligned}$$

Pappus. The area is $\int_0^1 (x - x^2) dx = \frac{1}{6}$, and by symmetry the centroid has $\bar{x} = \frac{1}{2}$, at distance $\frac{3}{2}$ from the axis. So $V = 2\pi \cdot \frac{3}{2} \cdot \frac{1}{6} = \frac{\pi}{2}$.

— **Problem 8.4** Length of a curve in y

Find the length of $x = \frac{y^4}{8} + \frac{1}{4y^2}$ for $1 \leq y \leq 2$.

Solution. Integrate in y : $L = \int \sqrt{1 + (dx/dy)^2} dy$. Here $\frac{dx}{dy} = \frac{y^3}{2} - \frac{1}{2y^3}$, and

$$1 + \left(\frac{y^3}{2} - \frac{1}{2y^3}\right)^2 = \frac{y^6}{4} + \frac{1}{2} + \frac{1}{4y^6} = \left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2.$$

The square root is exact, as designed: $L = \int_1^2 \left(\frac{y^3}{2} + \frac{1}{2y^3}\right) dy = \left[\frac{y^4}{8} - \frac{1}{4y^2}\right]_1^2 = \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right) = \frac{33}{16}$.

— **Problem 8.5** Winding up a cable

A 60 m cable weighing 3 N/m hangs from the top of a building, with a 500 N load at its end. How much work winds the cable and load to the top?

Solution. The load is lifted 60 m: $500 \cdot 60 = 30\,000$ J. For the cable, the piece at distance y below the top, of length Δy , weighs $3\Delta y$ and is lifted y : $\int_0^{60} 3y dy = \frac{3}{2} \cdot 3600 = 5400$ J. Total 35 400 J. Equivalently, the cable's weight 180 N acts at its center of mass, 30 m down: $180 \cdot 30 = 5400$ J.

— **Problem 8.6** Surface area of a paraboloid dish

Find the area of the surface obtained by revolving $y = \sqrt{x}$, $0 \leq x \leq 4$, about the x -axis.

Solution. $y\sqrt{1+y'^2} = \sqrt{x}\sqrt{1+\frac{1}{4x}} = \sqrt{x+\frac{1}{4}}$. So $S = \int_0^4 2\pi\sqrt{x+\frac{1}{4}} dx = \frac{4\pi}{3} \left[\left(x+\frac{1}{4}\right)^{3/2}\right]_0^4 = \frac{4\pi}{3} \left(\frac{17\sqrt{17}}{8} - \frac{1}{8}\right) = \frac{\pi}{6}(17\sqrt{17}-1) \approx 36.18$. The same number appears in Volume IV as the area of the paraboloid $z = x^2 + y^2$ below $z = 4$, computed as a surface integral: the two surfaces are congruent.

— **Problem 8.7** Centroid of a semicircular region and of a semicircular wire

Find the centroid of the half-disk $x^2 + y^2 \leq r^2$, $y \geq 0$, and of the semicircular arc $x^2 + y^2 = r^2$, $y \geq 0$. Use Pappus to check.

Solution. *Region.* By the formula with $f = \sqrt{r^2 - x^2}$, $A = \frac{\pi r^2}{2}$: $\bar{y} =$

$\frac{2}{\pi r^2} \int_{-r}^r \frac{1}{2}(r^2 - x^2)dx = \frac{2}{\pi r^2} \cdot \frac{2r^3}{3} = \frac{4r}{3\pi}$, as for the quarter disk. Check: revolving the half-disk about the x -axis gives a ball, $V = 2\pi \cdot \frac{4r}{3\pi} \cdot \frac{\pi r^2}{2} = \frac{4}{3}\pi r^3$.

Arc. For a curve, $\bar{y} = \frac{1}{L} \int_L y ds$. On the semicircle $y ds = r dx$ (as in the hat-box example), so $\bar{y} = \frac{1}{\pi r} \int_{-r}^r r dx = \frac{2r}{\pi} \approx 0.637r$, higher than the region's $0.424r$, since the arc's mass is all at the rim. Pappus for curves (surface area = $2\pi\bar{y}L$) gives the sphere's area: $2\pi \cdot \frac{2r}{\pi} \cdot \pi r = 4\pi r^2$.

EXERCISES

A. COMPUTATION

- 8.1** Find the area between $y = x^3$ and $y = x$.
- 8.2** Find the area between $y = e^x$, $y = e^{-x}$, and $x = 1$.
- 8.3** Find the area enclosed by $x = y^2 - 4y$ and $x = 2y - y^2$.
- 8.4** Find the area enclosed by $y = \frac{1}{x}$, $y = x$, and $y = \frac{x}{4}$ in the first quadrant.
- 8.5** Find the volume when the region under $y = \sqrt{x}$, $0 \leq x \leq 4$, is revolved about the x -axis.
- 8.6** Find the volume when the region between $y = x^2$ and $y = 4$ is revolved about the y -axis, by disks and by shells.
- 8.7** Find the volume when the region between $y = x^2$ and $y = 2x$ is revolved about the x -axis.
- 8.8** Find the volume when the region between $y = x^2$ and $y = 2x$ is revolved about the line $x = -1$.
- 8.9** Find the volume of a cone of radius r and height h by revolving a triangle.
- 8.10** The base of a solid is the region between $y = x^2$ and $y = 1$; cross-sections perpendicular to the y -axis are squares. Find the volume.
- 8.11** Find the volume of the solid obtained by revolving $y = \sin x$, $0 \leq x \leq \pi$, about the x -axis.
- 8.12** Find the volume when the region under $y = e^{-x^2}$, $0 \leq x \leq 1$, is revolved about the y -axis.
- 8.13** Find the length of $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \leq x \leq 2$.
- 8.14** Find the length of $y = \ln(\cos x)$, $0 \leq x \leq \frac{\pi}{4}$.

- 8.15** Set up, but do not evaluate, the integral for the length of $y = x^2$ from 0 to 1, and estimate it with the trapezoid rule, $n = 4$.
- 8.16** Find the area of the surface obtained by revolving $y = x^3, 0 \leq x \leq 1$, about the x -axis.
- 8.17** Find the surface area of a cone of radius r and slant height ℓ by revolving a segment.
- 8.18** A force of 10 N stretches a spring 4 cm. How much work stretches it 10 cm from natural length?
- 8.19** A cylindrical tank of radius 2 m and height 5 m is full of water. Find the work to pump the water out over the top.
- 8.20** A hemispherical bowl of radius 1 m is full of water. Find the work to pump the water out over the rim.
- 8.21** A vertical rectangular gate 4 m wide and 2 m tall has its top edge 3 m below the water surface. Find the force on it.
- 8.22** A triangular plate with base 2 m and height 3 m is submerged vertically, base at the surface. Find the hydrostatic force on one side.
- 8.23** Find the centroid of the region under $y = \sin x, 0 \leq x \leq \pi$.
- 8.24** Find the centroid of the region between $y = x^2$ and $y = \sqrt{x}$.
- 8.25** Use Pappus to find the volume obtained by revolving the triangle with vertices $(1, 0), (3, 0), (2, 2)$ about the y -axis.
- 8.26** For the density $f(x) = kx(4 - x)$ on $[0, 4]$ (zero elsewhere), find k , the mean, and $P(X < 1)$.
- 8.27** The lifetime of a bulb has density $f(t) = 0.001e^{-0.001t}$ hours. Find the probability that it lasts more than 1500 hours, and its median lifetime.
- 8.28** Find the average value of $f(x) = xe^{-x^2}$ on $[0, 2]$.
- 8.29** Find the volume of the solid of revolution generated by $x^2 + (y - 3)^2 \leq 1$ about the x -axis, using Pappus.
- 8.30** A 10 kg bucket is lifted from the ground to a height of 12 m at constant speed with a rope weighing 0.8 kg/m. Initially it holds 36 kg of water, which leaks at a constant rate and is gone when the bucket reaches 12 m. Find the work done (use $g = 9.8$).

B. THEORY AND PROOF

- 8.31** Derive the formula for the volume of a spherical cap of height h cut from a ball of radius R : $V = \pi h^2 \left(R - \frac{h}{3}\right)$.
- 8.32** Show that the shell formula and the washer formula give the same volume when the region under $y = f(x)$, $0 \leq x \leq a$, with f decreasing, $f(a) = 0$, is revolved about the y -axis. (Integrate by parts, or use the inverse function.)
- 8.33** Prove that the centroid of a triangle is at the intersection of its medians, two-thirds of the way from each vertex.
- 8.34** Prove that the work required to pump out a tank equals the total weight of the liquid times the distance its center of mass must be raised.
- 8.35** Prove that if f is a probability density symmetric about μ (and has finite mean), its mean and median both equal μ .
- 8.36** Show that the arc length of $y = f(x)$ on $[a, b]$ is at least $\sqrt{(b-a)^2 + (f(b) - f(a))^2}$.

C. CHALLENGE

- 8.37** A ball of radius R has a cylindrical hole of length h drilled through its center. Show that the volume remaining depends only on h , not on R . (The *napkin-ring problem*.)
- 8.38** Find the volume of the solid common to two cylinders of radius r whose axes meet at right angles. (Archimedes' *bicylinder*; the cross-sections are squares.)
- 8.39** Show that one arch of $y = \sin x$, $0 \leq x \leq \pi$, has the same length as the ellipse $x^2 + \frac{y^2}{2} = 1$ has in one half-turn, by writing both as $\int_0^\pi \sqrt{1 + \cos^2 t} dt$, and estimate this length numerically.
- 8.40** A spherical tank of radius 5 m is half full of oil of density 900 kg/m^3 . Find the work to pump the oil out through a hole at the top.
- 8.41** Find the curve through $(0, 1)$ whose arc length from $x = 0$ to $x = t$ equals the area under it from 0 to t for every $t > 0$.

CHAPTER 9

TECHNIQUES OF INTEGRATION

LEARNING OBJECTIVES

- Integrate by parts, including repeated and cyclic cases, and derive reduction formulas.
- Integrate products of powers of trigonometric functions.
- Use trigonometric and hyperbolic substitutions for integrands containing $\sqrt{a^2 \pm x^2}$ and $\sqrt{x^2 - a^2}$.
- Integrate every rational function by partial fractions.
- Approximate integrals by the midpoint, trapezoid, and Simpson rules, with error bounds.
- Define, evaluate, and test for convergence improper integrals of both kinds.

The Fundamental Theorem reduces integration to antidifferentiation, but antidifferentiation, unlike differentiation, has no algorithm that always works. The derivative of any elementary function is elementary; the antiderivative of e^{-x^2} or $\frac{\sin x}{x}$ is not. This chapter assembles the methods that succeed for the large and important class of integrals that do have elementary antiderivatives, and the numerical methods for those that do not. Computer algebra systems now perform these techniques automatically, but knowing them is still essential: to recognize which integrals can be done at all, to put answers in useful form, and to understand the structure of the results, such as why logarithms and arctangents appear.

9.1 INTEGRATION BY PARTS

The product rule $(uv)' = u'v + uv'$, integrated, gives:

Theorem 9.1 (Integration by parts).

$$\int u \, dv = uv - \int v \, du, \quad \int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du.$$

The method trades the integral of $u \, dv$ for that of $v \, du$. It succeeds when u becomes simpler on differentiation and dv can be integrated. A useful guide for choosing u is the order LIATE: *Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential*. Choose u from as early in the list as possible.

Example 9.2 (Basic uses). Evaluate (a) $\int x e^x \, dx$, (b) $\int \ln x \, dx$, (c) $\int \arctan x \, dx$, (d) $\int x^2 \sin x \, dx$.

Solution. (a) $u = x, dv = e^x dx$: $x e^x - \int e^x dx = (x - 1)e^x + C$.

(b) $u = \ln x, dv = dx$: $x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$. The trick of taking $dv = dx$ works for any function whose derivative is simpler.

(c) $u = \arctan x, dv = dx$: $x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$.

(d) Twice: $u = x^2, dv = \sin x \, dx$ gives $-x^2 \cos x + \int 2x \cos x \, dx$; then $u = 2x, dv = \cos x \, dx$ gives $2x \sin x - \int 2 \sin x \, dx$. Altogether $-x^2 \cos x + 2x \sin x + 2 \cos x + C$. Repeated parts with a polynomial u is organized efficiently in the *tabular method*: differentiate u down to 0, integrate dv alongside, and combine diagonally with alternating signs.

Example 9.3 (A cyclic integral). Evaluate $\int e^x \cos x \, dx$.

Solution. Parts with $u = \cos x$: $I = e^x \cos x + \int e^x \sin x \, dx$. Parts again with $u = \sin x$: $\int e^x \sin x \, dx = e^x \sin x - I$. So $I = e^x \cos x + e^x \sin x - I$, and $I = \frac{1}{2} e^x (\sin x + \cos x) + C$. The original integral reappeared and was solved for algebraically. (Using complex exponentials, $e^x \cos x = \operatorname{Re} e^{(1+i)x}$, whose antiderivative is $\operatorname{Re} \frac{e^{(1+i)x}}{1+i}$, gives the same result directly.)

Example 9.4 (A reduction formula). Show that $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$, and evaluate $\int_0^{\pi/2} \sin^n x \, dx$.

Solution. Parts with $u = \sin^{n-1} x, dv = \sin x \, dx$: $I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$. Replace $\cos^2 x = 1 - \sin^2 x$: $I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$, and solve for I_n .

On $[0, \frac{\pi}{2}]$ the boundary term vanishes, so $J_n = \frac{n-1}{n} J_{n-2}$, with $J_0 = \frac{\pi}{2}$ and

$J_1 = 1$. Hence

$$J_{2m} = \frac{(2m-1)(2m-3)\cdots 1}{(2m)(2m-2)\cdots 2} \cdot \frac{\pi}{2}, \quad J_{2m+1} = \frac{(2m)(2m-2)\cdots 2}{(2m+1)(2m-1)\cdots 3}.$$

These are *Wallis' integrals*. Since $J_{2m+1} \leq J_{2m} \leq J_{2m-1}$ and the ratio of the outer two tends to 1, they yield Wallis' product $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$ (Exercise 9.33).

9.2 TRIGONOMETRIC INTEGRALS

To integrate $\sin^m x \cos^n x$:

- If m or n is odd, save one factor of the odd power, convert the rest with $\sin^2 + \cos^2 = 1$, and substitute $u = \cos x$ (if m is odd) or $u = \sin x$ (if n is odd).
- If both are even, reduce the powers with $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, and $\sin x \cos x = \frac{1}{2} \sin 2x$.

For $\tan^m x \sec^n x$: if n is even, save $\sec^2 x$ and use $u = \tan x$; if m is odd, save $\sec x \tan x$ and use $u = \sec x$. Two integrals are worth memorizing:

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C,$$

$$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C.$$

The first is verified by differentiating; the second follows from parts with $u = \sec x$, $dv = \sec^2 x \, dx$, and the cyclic trick.

Example 9.5 (Odd and even powers). Evaluate (a) $\int \sin^3 x \cos^2 x \, dx$, (b) $\int_0^\pi \sin^4 x \, dx$, (c) $\int \tan^3 x \sec^4 x \, dx$.

Solution. (a) $\sin^3 x = (1 - \cos^2 x) \sin x$; with $u = \cos x$: $-\int (1 - u^2)u^2 du = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$.

(b) $\sin^4 x = \left(\frac{1 - \cos 2x}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) = \frac{1}{4}\left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)$. Over $[0, \pi]$ the cosines integrate to 0, leaving $\frac{3}{8}\pi$. (This is $2J_4$ from Example 9.4: $2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$.)

(c) Save $\sec^2 x$: $\tan^3 x \sec^2 x \cdot \sec^2 x = \tan^3 x (1 + \tan^2 x) \sec^2 x$. With $u = \tan x$: $\int (u^3 + u^5) du = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$.

Example 9.6 (Products of sines and cosines). For integers $m, n \geq 1$, show that $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi$ if $m = n$ and 0 otherwise.

Solution. Use $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$. If $m \neq n$, both $\cos(m - n)x$ and $\cos(m + n)x$ integrate to zero over a full period. If $m = n$, the first term is $\frac{1}{2} \cos 0 = \frac{1}{2}$, integrating to π . These *orthogonality relations* are the foundation of Fourier series, developed in Volume IV.

9.3 TRIGONOMETRIC SUBSTITUTION

Square roots of quadratics are removed by substitutions that turn them into perfect squares through a Pythagorean identity:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

The restricted ranges make the substitutions invertible and fix the signs of the square roots. To return to x , draw the reference triangle: for $x = a \sin \theta$, the opposite side is x , the hypotenuse a , and the adjacent side $\sqrt{a^2 - x^2}$.

Example 9.7 (The area of an ellipse). Find the area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. By symmetry $A = 4 \cdot \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx$. With $x = a \sin \theta$, $dx = a \cos \theta \, d\theta$, and limits 0 to $\frac{\pi}{2}$:

$$\int_0^a \sqrt{a^2 - x^2} \, dx = \int_0^{\pi/2} a^2 \cos^2 \theta \, d\theta = a^2 \cdot \frac{\pi}{4}.$$

So $A = 4 \cdot \frac{b}{a} \cdot \frac{\pi a^2}{4} = \pi ab$. For $a = b = r$ this is πr^2 : a proof that the area of a circle is πr^2 , given the definition of π as half the circumference of the unit circle (which entered through the radian measure of θ).

Example 9.8 (A secant substitution). Evaluate $\int \frac{dx}{x^2 \sqrt{x^2 + 4}}$ and $\int \frac{\sqrt{x^2 - 9}}{x} \, dx$ ($x > 3$).

Solution. For the first, $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{x^2 + 4} = 2 \sec \theta$:

$$\int \frac{2 \sec^2 \theta d\theta}{4 \tan^2 \theta \cdot 2 \sec \theta} = \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = -\frac{1}{4 \sin \theta} + C = -\frac{\sqrt{x^2 + 4}}{4x} + C,$$

reading $\sin \theta = \frac{x}{\sqrt{x^2 + 4}}$ from the triangle with opposite side x and adjacent side 2.

For the second, $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 9} = 3 \tan \theta$:
 $\int \frac{3 \tan \theta}{3 \sec \theta} 3 \sec \theta \tan \theta d\theta = 3 \int \tan^2 \theta d\theta = 3(\tan \theta - \theta) + C = \sqrt{x^2 - 9} - 3 \operatorname{arcsec} \frac{x}{3} + C.$

Completing the square brings any quadratic to one of these forms: $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{9-(x-2)^2}} = \arcsin \frac{x-2}{3} + C$. Hyperbolic substitutions, $x = a \sinh t$ for $\sqrt{a^2 + x^2}$ and $x = a \cosh t$ for $\sqrt{x^2 - a^2}$, are an alternative that leads to the logarithmic forms directly: $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C = \ln(x + \sqrt{x^2 + a^2}) + C'$.

9.4 PARTIAL FRACTIONS

Every rational function $\frac{P(x)}{Q(x)}$ can be integrated in elementary terms, and the method is the algebra of partial fractions. If $\deg P \geq \deg Q$, first divide to write $\frac{P}{Q} = S(x) + \frac{R(x)}{Q(x)}$ with $\deg R < \deg Q$. Then factor Q into linear factors and irreducible quadratics (always possible over $\mathbb{R}$, by the Fundamental Theorem of Algebra), and decompose:

- each factor $(x - a)^k$ contributes $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$;
- each irreducible factor $(x^2 + bx + c)^k$ contributes $\frac{B_1 x + C_1}{x^2 + bx + c} + \cdots + \frac{B_k x + C_k}{(x^2 + bx + c)^k}.$

The constants are found by multiplying through by Q and comparing coefficients, or by substituting convenient values of x . Each piece integrates to a rational function, a logarithm, or an arctangent. Hence:

Theorem 9.9. *The antiderivative of every rational function is a sum of a rational function, logarithms of linear and quadratic polynomials, and arctangents of linear functions.*

Example 9.10 (Distinct linear factors). Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$.

Solution. $2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2)$. Write $\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$, so $x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$. Substituting the roots: $x = 0$ gives $-1 = -2A$, $A = \frac{1}{2}$; $x = \frac{1}{2}$ gives $\frac{1}{4} = \frac{5}{4}B$, $B = \frac{1}{5}$; $x = -2$ gives $-1 = 10C$, $C = -\frac{1}{10}$. So the integral is $\frac{1}{2} \ln|x| + \frac{1}{10} \ln|2x - 1| - \frac{1}{10} \ln|x + 2| + C$. (Substituting roots, the *cover-up method*, finds each coefficient of a simple linear factor in one step.)

Example 9.11 (Repeated and quadratic factors). Evaluate $\int \frac{2x^2-x+4}{x^3+4x} dx$ and $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

Solution. First: $x^3 + 4x = x(x^2 + 4)$, and $\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$. Then $2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$: $A = 1$ (from $x = 0$), $A + B = 2$ so $B = 1$, and $C = -1$. The integral is $\ln|x| + \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \arctan \frac{x}{2} + C$. Second: divide first, $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$, and $x^3 - x^2 - x + 1 = (x - 1)^2(x + 1)$. Then $\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$ with $4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$: $x = 1$ gives $B = 2$; $x = -1$ gives $C = -1$; the x^2 coefficient gives $A + C = 0$, so $A = 1$. The integral is $\frac{x^2}{2} + x + \ln|x - 1| - \frac{2}{x-1} - \ln|x + 1| + C$.

Example 9.12 (A rationalizing substitution). Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.

Solution. Let $u = \sqrt{x+4}$, so $x = u^2 - 4$ and $dx = 2u du$: $\int \frac{2u^2}{u^2-4} du = \int \left(2 + \frac{8}{u^2-4}\right) du = 2u + 2 \ln \left| \frac{u-2}{u+2} \right| + C$, using $\frac{8}{u^2-4} = \frac{2}{u-2} - \frac{2}{u+2}$. Substitute back $u = \sqrt{x+4}$. The substitution $t = \tan \frac{x}{2}$ likewise converts any rational function of $\sin x$ and $\cos x$ into a rational function of t (Exercise 9.34).

9.4.1 STRATEGY

No single method works for everything. A reasonable order: simplify the integrand algebraically; look for an obvious substitution (is the derivative of part of the integrand also present?); classify by form (rational function, trigonometric powers, square root of a quadratic, product of unlike functions for parts); try again with a different manipulation. And recognize when to stop: integrals such as $\int e^{-x^2} dx$, $\int \frac{\sin x}{x} dx$, $\int \sqrt{1+x^3} dx$, $\int \frac{dx}{\ln x}$ have no elementary antiderivatives (a theorem of Liouville from the 1830s), and they must be handled numerically or

by series (Chapter 13).

9.5 NUMERICAL INTEGRATION

On $[a, b]$ with n subintervals of width $h = \frac{b-a}{n}$ and points $x_i = a + ih$, three rules approximate $\int_a^b f$:

$$\begin{aligned} M_n &= h[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad (\bar{x}_i \text{ the midpoints}), \\ T_n &= \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)], \\ S_n &= \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)], \end{aligned}$$

where n must be even for Simpson's rule. The trapezoid rule replaces f on each subinterval by a line; Simpson's rule replaces it on each pair of subintervals by the parabola through three points.

Theorem 9.13 (Error bounds). If $|f''| \leq K$ on $[a, b]$, then $|E_T| \leq \frac{K(b-a)^3}{12n^2}$ and $|E_M| \leq \frac{K(b-a)^3}{24n^2}$. If $|f^{(4)}| \leq K$, then $|E_S| \leq \frac{K(b-a)^5}{180n^4}$.

Doubling n divides the trapezoid and midpoint errors by about 4, and Simpson's by about 16. Simpson's rule is exact for cubics, since $f^{(4)} = 0$, one degree more than its construction from parabolas suggests.

Example 9.14 (Three rules for $\ln 2$). Approximate $\int_1^2 \frac{dx}{x} = \ln 2 = 0.69314718$.

Solution.

n	M_n	T_n	S_n	error T_n	error S_n
4	0.691220	0.697024	0.693254	-3.9×10^{-3}	-1.1×10^{-4}
8	0.692661	0.694122	0.693155	-9.7×10^{-4}	-7.4×10^{-6}
16	0.693025	0.693391	0.693148	-2.4×10^{-4}	-4.7×10^{-7}

The ratios of errors match Theorem 9.13: 4 for the trapezoid rule and 16 for Simpson's. The midpoint error is about half the trapezoid error and of opposite sign, a pattern explained by the theory: $S_{2n} = \frac{2M_n + T_n}{3}$ cancels their leading errors. For $n = 8$ the bounds, with $K = \max |f''| = 2$ and $K = \max |f^{(4)}| = 24$ on $[1, 2]$, are 2.6×10^{-3} and 3.3×10^{-5} ; the actual errors are well within them.

9.6 IMPROPER INTEGRALS

The definite integral was defined for bounded functions on bounded intervals. Integrals over infinite intervals or of unbounded functions are defined as limits.

Definition 9.15 (Improper integrals). *Type 1.* $\int_a^\infty f \, dx = \lim_{t \rightarrow \infty} \int_a^t f \, dx$, and similarly for $\int_{-\infty}^b$. $\int_{-\infty}^\infty f = \int_{-\infty}^c f + \int_c^\infty f$, where both must converge.

Type 2. If f is unbounded near b , $\int_a^b f \, dx = \lim_{t \rightarrow b^-} \int_a^t f \, dx$; similarly at a , or at an interior point (split there).

The integral *converges* if the limit exists as a finite number, and *diverges* otherwise.

Theorem 9.16 (p -integrals). $\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p > 1$, to $\frac{1}{p-1}$.
 $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$, to $\frac{1}{1-p}$.

Proof. For $p \neq 1$, $\int_1^t x^{-p} dx = \frac{t^{1-p}-1}{1-p}$, which converges as $t \rightarrow \infty$ iff $1-p < 0$. For $p = 1$, $\int_1^t \frac{dx}{x} = \ln t \rightarrow \infty$. The second case is similar with $t \rightarrow 0^+$. ■

The borderline $\frac{1}{x}$ diverges at both ends. Functions must decay faster than $\frac{1}{x}$ at infinity, and blow up more slowly than $\frac{1}{x}$ at 0, to be integrable.

Example 9.17 (Improper integrals). Evaluate or show divergence: (a) $\int_0^\infty e^{-x} dx$, (b) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$, (c) $\int_0^1 \ln x \, dx$, (d) $\int_0^3 \frac{dx}{x-1}$.

Solution. (a) $\lim_{t \rightarrow \infty} (1 - e^{-t}) = 1$. (b) $\lim \arctan t - \lim \arctan(-t) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$. (c) $\lim_{t \rightarrow 0^+} [x \ln x - x]_t^1 = -1 - \lim (t \ln t - t) = -1$, using $t \ln t \rightarrow 0$. (d) Unbounded at 1: split. $\int_0^1 \frac{dx}{x-1} = \lim_{t \rightarrow 1^-} \ln |t-1| = -\infty$, so the integral diverges; there is no need to examine $\int_1^3$. Ignoring the singularity and writing $\ln |x-1| \Big|_0^3 = \ln 2$ would be wrong.

Theorem 9.18 (Comparison test). Suppose $0 \leq f(x) \leq g(x)$ for $x \geq a$. If $\int_a^\infty g$ converges, so does $\int_a^\infty f$; if $\int_a^\infty f$ diverges, so does $\int_a^\infty g$.

Proof. $F(t) = \int_a^t f$ increases with t and is bounded above by $\int_a^\infty g$. A bounded

increasing function has a limit at infinity, by the completeness axiom: the limit is $\sup F$. ■

So $\int_1^\infty e^{-x^2} dx$ converges, since $e^{-x^2} \leq e^{-x}$ for $x \geq 1$; and $\int_1^\infty \frac{1+e^{-x}}{x} dx$ diverges, since the integrand exceeds $\frac{1}{x}$. A refinement, the *limit comparison test*, compares $\frac{f}{g} \rightarrow L \in (0, \infty)$: then both converge or both diverge. It shows at once that $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges, by comparison with $x^{-3/2}$.

Example 9.19 (The Gamma function). For $s > 0$, $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ converges, and $\Gamma(n+1) = n!$.

Solution. Near 0, $x^{s-1} e^{-x} \leq x^{s-1}$, integrable for $s > 0$ by Theorem 9.16. Near infinity, $x^{s-1} e^{-x} \leq e^{-x/2}$ for large x , since $x^{s-1} e^{-x/2} \rightarrow 0$ (Theorem 6.3). Integrating by parts, $\Gamma(s+1) = [-x^s e^{-x}]_0^\infty + s \int_0^\infty x^{s-1} e^{-x} dx = s\Gamma(s)$, and $\Gamma(1) = 1$. By induction $\Gamma(n+1) = n!$. The Gamma function extends the factorial to non-integers: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$.

9.7 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 9.1 Parts and substitution together

Evaluate $\int_0^1 \arcsin x \, dx$ and $\int e^{\sqrt{x}} dx$.

Solution. First, parts with $u = \arcsin x$, $dv = dx$: $[x \arcsin x]_0^1 - \int_0^1 \frac{x \, dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + [\sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1$. Geometrically, this is the area of the region left of $y = \arcsin x$'s inverse, $x = \sin y$, subtracted from a rectangle: $\frac{\pi}{2} \cdot 1 - \int_0^{\pi/2} \sin y \, dy$.

Second, substitute $t = \sqrt{x}$, $x = t^2$, $dx = 2t \, dt$: $\int 2te^t \, dt = 2(t-1)e^t + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C$.

— Problem 9.2 An integral of a square root, three ways

Evaluate $\int_0^1 \sqrt{1+x^2} \, dx$.

Solution. With $x = \tan \theta$: $\int_0^{\pi/4} \sec^3 \theta \, d\theta = \frac{1}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_0^{\pi/4} = \frac{1}{2} (\sqrt{2} + \ln(1 + \sqrt{2})) \approx 1.1478$.

With $x = \sinh t$,

$$\int_0^{\sinh^{-1} 1} \cosh^2 t \, dt = \frac{1}{2} [t + \sinh t \cosh t]_0^{\sinh^{-1} 1} = \frac{1}{2} (\ln(1 + \sqrt{2}) + \sqrt{2}),$$

the same, since $\sinh^{-1} 1 = \ln(1 + \sqrt{2})$ and $\cosh(\sinh^{-1} 1) = \sqrt{2}$.

Numerically, Simpson's rule with $n = 4$ gives 1.14778, within about 10^{-5} . This integral is the arc length of the parabola $y = \frac{x^2}{2}$ from 0 to 1.

— Problem 9.3 Partial fractions with a quadratic

Evaluate $\int_0^1 \frac{dx}{x^3+1}$.

Solution. $x^3 + 1 = (x + 1)(x^2 - x + 1)$, with irreducible quadratic factor. Write $\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$. At $x = -1$: $1 = 3A$, $A = \frac{1}{3}$. Comparing x^2 coefficients, $A + B = 0$, $B = -\frac{1}{3}$; constants, $A + C = 1$, $C = \frac{2}{3}$. So the quadratic part is $-\frac{1}{3} \cdot \frac{x-2}{x^2-x+1} = -\frac{1}{6} \cdot \frac{2x-1}{x^2-x+1} + \frac{1}{2} \cdot \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}$, splitting the numerator into a multiple of the derivative of the denominator plus a constant. Integrating,

$$\begin{aligned} \int_0^1 \frac{dx}{x^3+1} &= \left[\frac{1}{3} \ln(x+1) - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} \right]_0^1 \\ &= \frac{1}{3} \ln 2 + \frac{\pi}{3\sqrt{3}} \approx 0.8356. \end{aligned}$$

— Problem 9.4 Convergence by comparison

Determine whether each converges: (a) $\int_1^\infty \frac{\sin^2 x}{x^2} dx$, (b) $\int_0^1 \frac{dx}{\sqrt{x+x^3}}$, (c) $\int_2^\infty \frac{dx}{\ln x}$, (d) $\int_0^\infty \frac{\sin x}{x} dx$.

Solution. (a) $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$, which is integrable on $[1, \infty)$: converges.

(b) Near 0, $\frac{1}{\sqrt{x+x^3}} \leq \frac{1}{\sqrt{x}}$, integrable on $[0, 1]$ ($p = \frac{1}{2} < 1$): converges.

(c) $\ln x < x$, so $\frac{1}{\ln x} > \frac{1}{x}$, whose integral diverges: diverges.

(d) The integrand is not of one sign, so comparison does not apply directly. Integrate by parts on $[1, t]$: $\int_1^t \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x} \right]_1^t - \int_1^t \frac{\cos x}{x^2} dx$. The boundary term converges, and $\left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2}$ is integrable, so the integral converges. (Near 0, $\frac{\sin x}{x} \rightarrow 1$ is bounded.) Its value is $\frac{\pi}{2}$, found in Volume IV by complex analysis. But $\int_0^\infty \left| \frac{\sin x}{x} \right| dx$ diverges: the convergence relies on cancellation, like an alternating series (Chapter 12).

— **Problem 9.5** A reduction formula for a power of $\ln$

Show that $\int_0^1 (\ln x)^n dx = (-1)^n n!$.

Solution. Let $I_n = \int_0^1 (\ln x)^n dx$, an improper integral of type 2. Parts with $u = (\ln x)^n$, $dv = dx$: $I_n = [x(\ln x)^n]_0^1 - n \int_0^1 (\ln x)^{n-1} dx$. The boundary term is 0 at 1, and at 0, $x(\ln x)^n \rightarrow 0$ (substitute $x = e^{-t}$: $t^n e^{-t} \rightarrow 0$). So $I_n = -nI_{n-1}$, with $I_0 = 1$, giving $(-1)^n n!$. Alternatively, $x = e^{-t}$ turns I_n into $(-1)^n \int_0^\infty t^n e^{-t} dt = (-1)^n \Gamma(n+1)$.

— **Problem 9.6** How many subintervals?

How large must n be for the trapezoid rule, and for Simpson's rule, to approximate $\int_0^1 e^{x^2} dx$ within 10^{-6} ?

Solution. $f''(x) = (2 + 4x^2)e^{x^2} \leq 6e$ on $[0, 1]$. The trapezoid bound $\frac{6e}{12n^2} \leq 10^{-6}$ requires $n^2 \geq \frac{e}{2} \times 10^6$, $n \geq 1166$. For Simpson, $f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2} \leq 76e$, and $\frac{76e}{180n^4} \leq 10^{-6}$ requires $n^4 \geq 1.148 \times 10^6$, so $n \geq 32.7$; take $n = 34$ (even). Simpson needs about thirty times fewer function evaluations. With $n = 34$, Simpson gives 1.462 651 97, within 3×10^{-7} of the true value 1.462 651 75.

— **Problem 9.7** Escape from a singularity

For which p does $\int_0^\infty \frac{dx}{x^p(1+x)}$ converge?

Solution. Split at 1. On $(0, 1]$, $\frac{1}{x^p(1+x)}$ is between $\frac{1}{2x^p}$ and $\frac{1}{x^p}$, so it converges iff $p < 1$. On $[1, \infty)$, it is between $\frac{1}{2x^{p+1}}$ and $\frac{1}{x^{p+1}}$, so it converges iff $p + 1 > 1$, that is, $p > 0$. Both ends together: $0 < p < 1$. (For $p = \frac{1}{2}$ the substitution $x = u^2$ gives $\int_0^\infty \frac{2du}{1+u^2} = \pi$. In general the value is $\frac{\pi}{\sin \pi p}$, as Volume IV shows by contour integration.)

EXERCISES

A. COMPUTATION

9.1 Evaluate $\int x \cos 3x dx$ and $\int x^2 e^{-x} dx$.

9.2 Evaluate $\int x^3 \ln x dx$ and $\int_1^e \ln^2 x dx$.

9.3 Evaluate $\int e^{2x} \sin 3x dx$.

9.4 Evaluate $\int_0^1 x \arctan x dx$.

- 9.5** Evaluate $\int \sin^5 x \, dx$ and $\int \cos^2 x \sin^2 x \, dx$.
- 9.6** Evaluate $\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx$ and $\int \tan^3 x \, dx$.
- 9.7** Evaluate $\int \sin 5x \cos 2x \, dx$.
- 9.8** Evaluate $\int \frac{x^2 \, dx}{\sqrt{9-x^2}}$.
- 9.9** Evaluate $\int_0^2 \frac{dx}{(4+x^2)^{3/2}}$.
- 9.10** Evaluate $\int \frac{dx}{\sqrt{x^2+6x+13}}$ and $\int \frac{dx}{x^2+6x+13}$.
- 9.11** Evaluate $\int \frac{5x+1}{(2x+1)(x-1)} dx$.
- 9.12** Evaluate $\int \frac{x^3+1}{x^2-4} dx$.
- 9.13** Evaluate $\int \frac{dx}{x^2(x+1)}$.
- 9.14** Evaluate $\int \frac{x^2+1}{x(x^2+3)} dx$.
- 9.15** Evaluate $\int_0^1 \frac{x \, dx}{(x+1)^2(x^2+1)}$.
- 9.16** Evaluate $\int \frac{dx}{1+\sqrt{x}}$ and $\int \frac{dx}{e^x+1}$.
- 9.17** Use T_4 , M_4 , and S_4 to approximate $\int_0^1 e^{-x^2} dx$; compare with 0.746824.
- 9.18** Find n guaranteeing that T_n approximates $\int_1^2 \frac{dx}{x}$ within 10^{-4} .
- 9.19** Evaluate or show divergence: $\int_1^\infty \frac{dx}{x^3}$, $\int_0^\infty x e^{-x} dx$, $\int_1^\infty \frac{dx}{\sqrt{x}}$.
- 9.20** Evaluate or show divergence: $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$, $\int_0^1 \frac{dx}{x^2}$, $\int_0^\infty \frac{x \, dx}{(1+x^2)^2}$.
- 9.21** Evaluate $\int_e^\infty \frac{dx}{x \ln^2 x}$ and determine whether $\int_e^\infty \frac{dx}{x \ln x}$ converges.
- 9.22** Evaluate $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$.
- 9.23** Use comparison to decide convergence: $\int_1^\infty \frac{dx}{x^2+\sqrt{x}}$, $\int_1^\infty \frac{2+\cos x}{x} dx$, $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$.
- 9.24** Find the area of the region under $y = \frac{1}{x^2+x}$ for $x \geq 1$.
- 9.25** Evaluate $\Gamma(5)$, $\Gamma(\frac{3}{2})$, and $\int_0^\infty x^5 e^{-2x} dx$.
- 9.26** Evaluate $\int \sec^3 x \, dx$ by parts, verifying the formula in the text.
- 9.27** Evaluate $\int x\sqrt{x+1} \, dx$ in two ways: by parts and by substitution.

9.28 Evaluate $\int \frac{dx}{x\sqrt{x^2-1}}$ for $x > 1$.

9.29 Evaluate $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$.

9.30 Evaluate $\int \frac{dx}{x^4-1}$.

B. THEORY AND PROOF

9.31 Derive the reduction formula $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$ and use it to find $\int x^3 e^x dx$.

9.32 Derive $\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$.

9.33 Using $J_{2m+1} \leq J_{2m} \leq J_{2m-1}$ from Example 9.4, prove Wallis' product $\frac{\pi}{2} = \lim_{m \rightarrow \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2m)(2m)}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2m-1)(2m+1)}$.

9.34 Show that the substitution $t = \tan \frac{x}{2}$ gives $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$, and use it to evaluate $\int \frac{dx}{1+\sin x + \cos x}$.

9.35 Prove that Simpson's rule is exact for cubic polynomials.

9.36 Prove that $S_{2n} = \frac{1}{3} T_n + \frac{2}{3} M_n$.

9.37 Prove that $\int_1^\infty \frac{\cos x}{x} dx$ converges.

9.38 Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ is equivalent to $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.

C. CHALLENGE

9.39 Evaluate $\int_0^{\pi/2} \ln(\sin x) dx$. (Show $I = \int_0^{\pi/2} \ln \cos x dx$, add, and use $\sin 2x = 2 \sin x \cos x$; the answer is $-\frac{\pi}{2} \ln 2$.)

9.40 Prove that if f is continuous and decreasing on $[1, \infty)$ and $\lim f = 0$, then $\int_1^\infty f(x) \sin x dx$ converges.

9.41 Compute $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ by long division, and use it to show $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} + \frac{1}{1260}$.

9.42 Find $\int_0^\infty \frac{dx}{(1+x^2)^n}$ for all $n \geq 1$ by a reduction formula.

9.43 Prove Stirling's lower bound $n! \geq \left(\frac{n}{e}\right)^n$ by comparing $\ln n! = \sum \ln k$ with $\int_1^n \ln x dx$.

CHAPTER 10

DIFFERENTIAL EQUATIONS

LEARNING OBJECTIVES

- Model growth, decay, cooling, mixing, and motion with differential equations, and verify solutions.
- Sketch direction fields and approximate solutions with Euler's method, estimating its error.
- Solve separable equations, including the exponential and logistic models, and orthogonal trajectories.
- Solve first-order linear equations with an integrating factor, and apply them to circuits and mixing.
- Find equilibria of autonomous equations and of predator–prey systems, and classify their stability.

A *differential equation* is an equation involving an unknown function and its derivatives. Newton's second law, the law of radioactive decay, the laws of population growth and of heat flow are all differential equations: they state how a quantity changes, and leave it to us to deduce what the quantity is. This chapter treats *first-order* equations, $y' = F(x, y)$, which involve only the first derivative. Second-order linear equations, the equations of oscillation, are studied in Volume IV.

10.1 MODELING AND DIRECTION FIELDS

A *solution* of $y' = F(x, y)$ on an interval is a differentiable function satisfying the equation there. The general solution usually contains an arbitrary constant, fixed by an *initial condition* $y(x_0) = y_0$; together they form an *initial value problem*. Under mild hypotheses (that F and $\partial F / \partial y$ are continuous), every initial value problem has exactly one solution near x_0 (the Picard–Lindelöf theorem, Volume

IV).

Example 10.1 (Verifying a solution). Show that $y = Ce^{x^2} - \frac{1}{2}$ solves $y' = 2xy + x$ for every C , and find the solution with $y(0) = 1$.

Solution. $y' = 2Cxe^{x^2}$ and $2xy + x = 2Cxe^{x^2} - x + x = 2Cxe^{x^2}$. They agree. Then $y(0) = C - \frac{1}{2} = 1$ gives $C = \frac{3}{2}$.

10.1.1 DIRECTION FIELDS

The equation $y' = F(x, y)$ specifies a slope at every point of the plane. Drawing a short segment with that slope at many points gives a *direction field*, and solution curves are the curves that are everywhere tangent to it. A direction field shows the qualitative behavior of all solutions at once, before any formula is found.

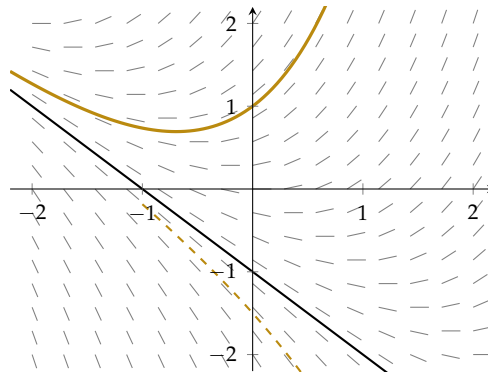


Figure 10.1: Direction field of $y' = x + y$, with the solution through $(0, 1)$ (gold), the straight-line solution $y = -x - 1$ (black), and the solution through $(0, -1.5)$ (dashed). Solutions above the line rise away from it; those below fall away.

10.1.2 EULER'S METHOD

Follow the direction field in small steps: from (x_n, y_n) , move a distance h to the right along the slope $F(x_n, y_n)$:

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hF(x_n, y_n).$$

This is linear approximation applied repeatedly. The error at a fixed x is proportional to h : Euler's method is *first order*.

Example 10.2 (Euler's method and its error). Approximate $y(1)$ for $y' = x + y$, $y(0) = 1$, whose exact solution is $y = 2e^x - x - 1$.

Solution. The exact value is $y(1) = 2e - 2 \approx 3.43656$.

h	Euler $y(1)$	error	error/ h
0.1	3.18748	0.24908	2.49
0.05	3.30660	0.12997	2.60
0.01	3.40963	0.02694	2.69

Halving h halves the error. The approximations fall short because the solution is convex, and each tangent step undershoots. Better methods, such as the Runge–Kutta method of Appendix C, reduce the error to order h^4 .

10.2 SEPARABLE EQUATIONS

An equation is *separable* if it can be written $\frac{dy}{dx} = g(x)h(y)$. Divide by $h(y)$ (where $h(y) \neq 0$) and integrate both sides:

$$\int \frac{dy}{h(y)} = \int g(x) dx.$$

This is justified by the chain rule: if $H'(y) = \frac{1}{h(y)}$, then $\frac{d}{dx}H(y(x)) = \frac{y'}{h(y)} = g(x)$, so $H(y) = G(x) + C$. Constant solutions $y = c$ with $h(c) = 0$, lost in the division, must be checked separately.

COMMON ERROR. Dividing by $h(y)$ silently discards the constant solutions where $h(y) = 0$. For $y' = y^2$, separation gives $y = -\frac{1}{x+C}$, and no value of C produces the solution $y \equiv 0$. Always check the zeros of h before dividing.

Example 10.3 (A separable equation). Solve $\frac{dy}{dx} = \frac{x^2}{y^2}$ with $y(0) = 2$, and $\frac{dy}{dx} = xy^2$ with $y(0) = 1$.

Solution. First: $\int y^2 dy = \int x^2 dx$, so $y^3 = x^3 + C$; with $y(0) = 2$, $C = 8$ and $y = \sqrt[3]{x^3 + 8}$.

Second: $\int y^{-2} dy = \int x dx$ gives $-\frac{1}{y} = \frac{x^2}{2} + C$, so $y = \frac{-2}{x^2 + 2C}$, together with the solution $y = 0$ lost by dividing. With $y(0) = 1$, $C = -\frac{1}{2}$, so $y = \frac{2}{2-x^2}$, which blows up at $x = \sqrt{2}$. A perfectly smooth equation can have solutions that cease to exist at a finite x : the interval of existence depends on the initial

condition.

10.2.1 EXPONENTIAL GROWTH AND DECAY

Theorem 10.4. The solutions of $\frac{dy}{dt} = ky$ are exactly $y = y_0 e^{kt}$.

Proof. $y_0 e^{kt}$ is a solution. Conversely, if $y' = ky$, then $(ye^{-kt})' = y'e^{-kt} - kye^{-kt} = 0$, so ye^{-kt} is constant, equal to $y(0)$. ■

The proof does not divide by y , so it covers solutions that might vanish. Growth at a rate proportional to size is the law of unchecked populations ($k > 0$), of radioactive decay ($k < 0$, with half-life $\frac{\ln 2}{|k|}$), and of continuously compounded interest.

Example 10.5 (Newton's law of cooling). An object's temperature changes at a rate proportional to the difference from the surroundings: $\frac{dT}{dt} = -k(T - T_s)$. A cup of coffee at 90°C is placed in a 20°C room and cools to 60°C in 10 minutes. When will it reach 30°C ?

Solution. With $u = T - 20$, $u' = -ku$, so $u = 70e^{-kt}$. At $t = 10$, $40 = 70e^{-10k}$, so $k = \frac{\ln(7/4)}{10} \approx 0.05596$ per minute. Then $10 = 70e^{-kt}$ gives $t = \frac{\ln 7}{k} \approx 34.8$ minutes. The temperature approaches 20°C exponentially and never reaches it.

10.2.2 THE LOGISTIC EQUATION

Real populations are limited by resources. The *logistic model* $\frac{dP}{dt} = kP(1 - \frac{P}{K})$ has growth rate nearly kP when P is small, falling to zero at the *carrying capacity* K . Separating with partial fractions, $\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$, gives

$$P(t) = \frac{K}{1 + Ae^{-kt}}, \quad A = \frac{K - P_0}{P_0}.$$

Every solution with $P_0 > 0$ tends to K : the equilibrium $P = K$ is stable. Growth is fastest when $P = \frac{K}{2}$, the inflection point of the S-shaped curve.

Example 10.6 (Fitting a logistic model). A population is 100 at $t = 0$ and 400 at $t = 5$, with carrying capacity 1000. Find the logistic model and the time at which the population reaches 900.

Solution. $A = \frac{1000-100}{100} = 9$. At $t = 5$: $400 = \frac{1000}{1+9e^{-5k}}$, so $9e^{-5k} = \frac{3}{2}$ and $k = \frac{\ln 6}{5} \approx 0.3584$. Then $900 = \frac{1000}{1+9e^{-kt}}$ gives $e^{-kt} = \frac{1}{81}$, $t = \frac{\ln 81}{k} = \frac{5 \ln 81}{\ln 6} \approx 12.26$.

10.2.3 ORTHOGONAL TRAJECTORIES

Curves that cross every member of a family at right angles are its *orthogonal trajectories*. In electrostatics, field lines are the orthogonal trajectories of equipotential curves; in fluid flow, streamlines are orthogonal to lines of constant potential. To find them, find the differential equation $y' = F(x, y)$ of the family, eliminating the parameter, and solve $y' = -\frac{1}{F(x, y)}$.

Example 10.7 (Parabolas and ellipses). Find the orthogonal trajectories of the parabolas $y = cx^2$.

Solution. $y' = 2cx$, and $c = \frac{y}{x^2}$, so $y' = \frac{2y}{x}$. The trajectories satisfy $y' = -\frac{x}{2y}$: $2y \, dy = -x \, dx$, so $y^2 = -\frac{x^2}{2} + C$, that is, the ellipses $\frac{x^2}{2} + y^2 = C$.

10.3 FIRST-ORDER LINEAR EQUATIONS

A *first-order linear* equation has the form $y' + P(x)y = Q(x)$. Multiply by the *integrating factor* $\mu(x) = e^{\int P \, dx}$. Since $\mu' = P\mu$, the left side becomes an exact derivative:

$$\mu y' + \mu P y = (\mu y)' = \mu Q, \quad \text{so} \quad y = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) \, dx + C \right).$$

Example 10.8 (An integrating factor). Solve $xy' + 2y = x^2 - x + 1$ for $x > 0$, with $y(1) = \frac{1}{2}$.

Solution. Divide by x : $y' + \frac{2}{x}y = x - 1 + \frac{1}{x}$. Then $\mu = e^{\int 2/x \, dx} = x^2$, and $(x^2 y)' = x^3 - x^2 + x$. Integrating, $x^2 y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$. With $y(1) = \frac{1}{2}$: $\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$, so $C = \frac{1}{12}$, and $y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{1}{12x^2}$.

Example 10.9 (An electric circuit). A circuit with resistance R ohms and inductance L henries, driven by a constant voltage E , carries a current satisfying $L \frac{dI}{dt} + RI = E$. Find $I(t)$ if $I(0) = 0$.

Solution. $I' + \frac{R}{L}I = \frac{E}{L}$, with $\mu = e^{Rt/L}$: $(Ie^{Rt/L})' = \frac{E}{L}e^{Rt/L}$, so $Ie^{Rt/L} = \frac{E}{R}e^{Rt/L} + C$, and $I(0) = 0$ gives $C = -\frac{E}{R}$. Hence $I(t) = \frac{E}{R}(1 - e^{-Rt/L})$.

The current rises toward the steady value $\frac{E}{R}$ given by Ohm's law, with *time constant* $\frac{L}{R}$: after one time constant it has reached $1 - e^{-1} \approx 63\%$ of its final value.

Example 10.10 (Mixing). A 1000 L tank holds pure water. Brine with 0.05 kg of salt per liter flows in at 10 L/min, and the well-mixed solution flows out at the same rate. Find the salt $y(t)$ in the tank.

Solution. Rate in: $0.05 \cdot 10 = 0.5$ kg/min. Rate out: the concentration $\frac{y}{1000}$ times 10 L/min. So $y' = 0.5 - \frac{y}{100}$, linear (and separable). With $y(0) = 0$: $y = 50(1 - e^{-t/100})$. After 60 minutes the tank holds $50(1 - e^{-0.6}) \approx 22.6$ kg, and as $t \rightarrow \infty$ the salt approaches 50 kg, the amount at the inflow concentration.

10.4 AUTONOMOUS EQUATIONS AND EQUILIBRIA

An equation $y' = f(y)$, with no explicit x or t , is *autonomous*. Its *equilibria* are the constant solutions, the zeros of f . Between equilibria f has constant sign, so solutions increase or decrease monotonically toward the neighboring equilibria, and by uniqueness can never cross them. An equilibrium c is *stable* if nearby solutions approach it, which holds when $f'(c) < 0$, and *unstable* when $f'(c) > 0$. The *phase line*, a number line marked with the equilibria and arrows showing the sign of f , summarizes the behavior of every solution.

Example 10.11 (A harvested population). A fish population obeys $P' = P(4 - P) - 3$ (logistic growth with constant harvesting, in suitable units). Find and classify the equilibria.

Solution. $f(P) = -P^2 + 4P - 3 = -(P - 1)(P - 3)$. Equilibria $P = 1$ and $P = 3$. Since $f'(P) = 4 - 2P$, $f'(1) = 2 > 0$ (unstable) and $f'(3) = -2 < 0$ (stable). Populations above 1 tend to 3; populations below 1 collapse to extinction in finite time. If the harvest rate rises above 4, the maximum of $P(4 - P)$, there are no equilibria and every population collapses: a model of overfishing.

10.4.1 PREDATOR–PREY SYSTEMS

Two interacting populations, prey $R(t)$ and predators $W(t)$, are modeled by the Lotka–Volterra equations

$$\frac{dR}{dt} = aR - bRW, \quad \frac{dW}{dt} = -cW + dRW,$$

with positive constants: prey grow without predators, predators die without prey, and encounters, at a rate proportional to RW , benefit the predators at the prey's expense. The equilibria are $(0, 0)$ and $(\frac{c}{d}, \frac{a}{b})$. Dividing the equations, $\frac{dW}{dR} = \frac{W(-c+dR)}{R(a-bW)}$, which is separable:

$$\int \left(\frac{a}{W} - b \right) dW = \int \left(-\frac{c}{R} + d \right) dR \implies a \ln W - bW + c \ln R - dR = K.$$

The trajectories in the RW -plane are closed curves around the nonzero equilibrium: the populations oscillate periodically, the predator peak lagging the prey peak. This was Volterra's 1926 explanation of fish-catch data from the Adriatic.

LOOKING AHEAD. Systems of several first-order equations, like the predator–prey model, are the general case: any higher-order equation can be rewritten as one. In Volume IV, linear systems $\mathbf{x}' = A\mathbf{x}$ are solved with eigenvalues, and the time-dependent Schrödinger equation is exactly such a system, of infinite dimension.

10.5 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 10.1 Terminal velocity

A skydiver of mass m falls with air resistance proportional to the square of speed: $m \frac{dv}{dt} = mg - kv^2$. Find $v(t)$ with $v(0) = 0$, and the terminal velocity.

Solution. Let $v_T = \sqrt{mg/k}$, where the forces balance. The equation becomes $\frac{dv}{dt} = g(1 - \frac{v^2}{v_T^2})$, separable:

$$\int \frac{dv}{1 - v^2/v_T^2} = \int g dt \implies v_T \tanh^{-1} \frac{v}{v_T} = gt,$$

using $\int \frac{du}{1-u^2} = \tanh^{-1} u$ (from Chapter 4). So $v(t) = v_T \tanh \frac{gt}{v_T}$, which tends to v_T . For a skydiver, $v_T \approx 55$ m/s, reached to within 1% after about $\frac{v_T}{g} \tanh^{-1}(0.99) \approx 14.9$ s. The hyperbolic tangent of Chapter 1 appears as the natural shape of this approach.

— **Problem 10.2** A linear equation with a periodic input

Solve $y' + y = \sin t$, $y(0) = 0$, and describe the long-term behavior.

Solution. $\mu = e^t$, and $(ye^t)' = e^t \sin t$. By the cyclic integration of Chapter 9, $\int e^t \sin t dt = \frac{1}{2}e^t(\sin t - \cos t)$. So $y = \frac{1}{2}(\sin t - \cos t) + Ce^{-t}$, and $y(0) = 0$ gives $C = \frac{1}{2}$:

$$y(t) = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}.$$

The term $\frac{1}{2}e^{-t}$ is a *transient* that dies out; the rest is the *steady state*, $\frac{1}{\sqrt{2}} \sin(t - \frac{\pi}{4})$, oscillating at the input frequency with amplitude reduced by $\frac{1}{\sqrt{2}}$ and phase delayed by $\frac{\pi}{4}$. Every linear system driven periodically behaves this way: the response forgets its initial condition and follows the input, scaled and shifted.

— **Problem 10.3** A tank that drains

Water drains from a cylindrical tank through a hole in the bottom. By Torricelli's law, $\frac{dh}{dt} = -k\sqrt{h}$. If the depth falls from 4 m to 1 m in 10 minutes, when is the tank empty?

Solution. Separating, $\int h^{-1/2} dh = -\int k dt$: $2\sqrt{h} = -kt + C$. With $h(0) = 4$, $C = 4$. With $h(10) = 1$: $2 = -10k + 4$, $k = \frac{1}{5}$. The tank is empty when $2\sqrt{h} = 0$: $t = \frac{C}{k} = 20$ minutes. Unlike exponential decay, which never reaches zero, the square root law empties the tank in finite time. The uniqueness theorem fails at $h = 0$, where $\sqrt{h}$ is not differentiable: the solution $h \equiv 0$ and the draining solution both pass through it, which is why running the equation backward from an empty tank cannot tell when it was full.

— **Problem 10.4** Exact solutions versus Euler

For $y' = y^2$, $y(0) = 1$, find the exact solution and show that Euler's method with any step size predicts finite values at $x = 1$, where the true solution is infinite.

Solution. Separating: $-\frac{1}{y} = x + C$, $C = -1$, so $y = \frac{1}{1-x}$, which blows up at $x = 1$. Euler's method computes $y_{n+1} = y_n + hy_n^2$, a finite number at every step: with $h = 0.1, 0.05$, and 0.01 it predicts $y(1) \approx 6.13, 9.55$, and 30.4 . The approximations grow as h shrinks but are finite for every h ; nothing in the output signals that the true solution does not exist at $x = 1$. Numerical methods can march straight

past a singularity without noticing it. Knowing from the exact solution, or from theory, where solutions exist is essential before trusting a computation.

— **Problem 10.5** Continuous compounding with deposits

An account earns 5% interest compounded continuously, and the owner deposits \$2000 per year continuously. Starting from zero, how much is in the account after 20 years? How much was deposited?

Solution. $A' = 0.05A + 2000$, linear: $(Ae^{-0.05t})' = 2000e^{-0.05t}$, so $Ae^{-0.05t} = -40\,000e^{-0.05t} + C$. With $A(0) = 0$, $C = 40\,000$, and $A(t) = 40\,000(e^{0.05t} - 1)$. At $t = 20$: $40\,000(e - 1) \approx \$68\,731$, of which \$40 000 was deposited and \$28 731 is interest.

EXERCISES

A. COMPUTATION

- 10.1** Verify that $y = \frac{\ln x + C}{x}$ solves $x^2y' + xy = 1$, and find the solution with $y(1) = 2$.
- 10.2** For which r does $y = e^{rx}$ satisfy $y'' + y' - 6y = 0$?
- 10.3** Sketch the direction field of $y' = y - x$, and the solutions through $(0, 0)$, $(0, 1)$, and $(0, 2)$.
- 10.4** Use Euler's method with $h = 0.2$ to approximate $y(1)$ for $y' = y$, $y(0) = 1$, and compare with e .
- 10.5** Use Euler's method with $h = 0.1$ to approximate $y(0.5)$ for $y' = x^2 + y^2$, $y(0) = 0$.
- 10.6** Solve $\frac{dy}{dx} = \frac{y}{x}$.
- 10.7** Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, $y(0) = 1$.
- 10.8** Solve $\frac{dy}{dx} = xe^{-y}$, $y(0) = 0$.
- 10.9** Solve $\frac{dy}{dt} = y \cos t$, $y(0) = 2$.
- 10.10** Solve $(1 + x^2)y' = xy$, $y(0) = 3$.
- 10.11** A radioactive isotope loses 20% of its mass in 5 years. Find its half-life.

- 10.12** A bacteria culture grows exponentially from 500 to 8000 in 4 hours. When does it reach 10^6 ?
- 10.13** A body at 80°C cools to 50°C in 20 minutes in a 20°C room. Find its temperature after 40 minutes.
- 10.14** Solve the logistic equation $P' = 0.2P(1 - P/500)$, $P(0) = 50$, and find when $P = 250$.
- 10.15** Find the orthogonal trajectories of $x^2 + y^2 = r^2$ and of $y = \frac{k}{x}$.
- 10.16** Solve $y' + 2y = e^{-x}$.
- 10.17** Solve $y' - \frac{y}{x} = x^2$ for $x > 0$, $y(1) = 0$.
- 10.18** Solve $y' + y \tan x = \cos^2 x$ for $|x| < \frac{\pi}{2}$.
- 10.19** Solve $t \frac{dy}{dt} + 3y = \frac{\sin t}{t^2}$ for $t > 0$.
- 10.20** An RC circuit satisfies $RQ' + \frac{Q}{C} = E$ with $R = 10$, $C = 0.01$, $E = 50$, $Q(0) = 0$. Find $Q(t)$ and the current $I = Q'$.
- 10.21** A tank contains 200 L of water with 10 kg of salt. Pure water enters at 4 L/min and the mixture leaves at 4 L/min. Find the salt after one hour.
- 10.22** A tank contains 100 L of pure water. Brine with 0.2 kg/L enters at 3 L/min and the mixture leaves at 2 L/min, so the volume increases. Find the salt $y(t)$.
- 10.23** Find and classify the equilibria of $y' = y^3 - y$, and sketch the phase line.
- 10.24** Find and classify the equilibria of $P' = P(P - 2)(5 - P)$.
- 10.25** For the predator-prey system $R' = 0.1R - 0.002RW$, $W' = -0.2W + 0.0004RW$, find the equilibria, and write the equation of the trajectories.
- 10.26** A ball of mass 1 kg falls with air resistance $0.5v$ N. Find $v(t)$ with $v(0) = 0$ and the terminal velocity ($g = 9.8$).
- 10.27** Solve the Bernoulli equation $y' + y = xy^3$ with the substitution $u = y^{-2}$.
- 10.28** Solve $\frac{dy}{dx} = \frac{x+y}{x}$ by writing it as a linear equation.
- 10.29** Solve the homogeneous equation $y' = \frac{y}{x} + \frac{x}{y}$ for $x > 0$ with the substitution $y = vx$.

- 10.30** An investment grows at 6% compounded continuously, with continuous withdrawals of \$3000 per year. What initial balance keeps the account level forever?

B. THEORY AND PROOF

- 10.31** Prove that if y_1 and y_2 solve $y' + P(x)y = Q(x)$, then $y_1 - y_2$ solves the homogeneous equation $y' + Py = 0$. Deduce the structure of the general solution.
- 10.32** Prove that the solutions of the logistic equation with $0 < P_0 < K$ are increasing and bounded by K , without solving the equation.
- 10.33** Show that $y = 0$ and $y = x^3$ both solve $y' = 3y^{2/3}$, $y(0) = 0$. Which hypothesis of the uniqueness theorem fails?
- 10.34** Show that the time for a logistic population to go from $\frac{K}{10}$ to $\frac{9K}{10}$ is $\frac{2\ln 9}{k}$, independent of K .
- 10.35** Show that the quantity $a \ln W - bW + c \ln R - dR$ is constant along solutions of the Lotka–Volterra system by differentiating it.
- 10.36** Show that the error of Euler's method for $y' = y$, $y(0) = 1$, at $x = 1$ is $e - (1 + h)^{1/h}$, and that it is approximately $\frac{eh}{2}$ for small h .

C. CHALLENGE

- 10.37** A mothball evaporates at a rate proportional to its surface area. Show that its radius decreases linearly, and find when it vanishes if half the volume disappears in one month.
- 10.38** A snowplow starts at noon, removing a fixed volume of snow per hour. Snow began falling before noon at a constant rate. The plow goes 2 km in the first hour and 1 km in the second. When did the snow start? (The classic problem of Agnew.)
- 10.39** Show that the curve whose tangent at every point P meets the x -axis at a distance from the foot of the ordinate equal to a constant a is an exponential curve. Find all such curves.
- 10.40** The Gompertz model $P' = kP \ln \frac{K}{P}$ describes tumor growth. Solve it with $u = \ln \frac{P}{K}$.
- 10.41** Show that every solution of $y' = \sin y$ with $0 < y(0) < \pi$ tends to π , and find the solutions explicitly.



PART III

CURVES AND SERIES

Parametric and polar curves, infinite sequences and series, and the representation of functions by power series.

CHAPTER 11

PARAMETRIC EQUATIONS AND POLAR COORDINATES

LEARNING OBJECTIVES

- Describe curves parametrically, eliminate parameters, and trace orientation.
- Find tangent lines, concavity, enclosed areas, arc lengths, and surface areas for parametric curves.
- Convert between rectangular and polar coordinates, and sketch polar curves.
- Compute slopes, areas, and lengths in polar coordinates.
- Write conic sections in focus–directrix form and use the polar equation of a conic to analyze orbits.

A curve in the plane is not always the graph of a function. A circle fails the vertical line test, and so does any curve that loops back and crosses itself. This chapter gives two more flexible descriptions: *parametric equations*, which give both coordinates as functions of a third variable, and *polar coordinates*, which locate a point by distance and direction. Both are the two-dimensional prototypes of the curves and coordinate systems of Volume IV.

11.1 PARAMETRIC CURVES

Definition 11.1. A *parametric curve* is a pair of continuous functions $x = f(t)$, $y = g(t)$ on an interval I . The variable t is the *parameter*; as t increases, the point $(f(t), g(t))$ traces the curve in a definite direction, its *orientation*.

A parametrization carries more information than the set of points it traces: it says

when each point is reached, in what order, and how often. The curves $(\cos t, \sin t)$ and $(\cos 2t, \sin 2t)$ for $0 \leq t \leq 2\pi$ trace the same circle, once and twice.

Example 11.2 (Eliminating the parameter). Identify the curves (a) $x = 1 + 2 \cos t$, $y = -1 + 3 \sin t$, $0 \leq t \leq 2\pi$; (b) $x = \sqrt{t}$, $y = 1 - t$, $t \geq 0$.

Solution. (a) $\cos t = \frac{x-1}{2}$ and $\sin t = \frac{y+1}{3}$, so $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$: an ellipse centered at $(1, -1)$, traced once counterclockwise starting from $(3, -1)$.

(b) $t = x^2$, so $y = 1 - x^2$ with $x \geq 0$: only the right half of the parabola, traced from $(0, 1)$ downward. Eliminating the parameter can lose the restriction on x ; the range of $x(t)$ must be recorded separately.

Example 11.3 (The cycloid). A wheel of radius r rolls along the x -axis without slipping. Find the path of a point on its rim that starts at the origin.

Solution. After the wheel turns through angle θ , it has rolled a distance $r\theta$ (no slipping), so its center is at $(r\theta, r)$. The point is at distance r from the center, at angle θ clockwise from straight down. Hence

$$x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta).$$

The *cycloid* has a cusp at each multiple of $2\pi r$, where the point touches the ground and is momentarily at rest.

The cycloid is the solution of two famous problems. It is the *brachistochrone*, the curve down which a bead slides from one point to another in the least time (Johann Bernoulli, 1696); and it is the *tautochrone*, the curve on which a bead released anywhere reaches the bottom in the same time (Huygens, 1673), which Huygens used to design an accurate pendulum clock.

11.2 CALCULUS WITH PARAMETRIC CURVES

11.2.1 TANGENTS

If $x = f(t)$ and $y = g(t)$ are differentiable and $f'(t) \neq 0$, then near t the curve is the graph of a function $y(x)$, and by the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

The tangent is horizontal where $\frac{dy}{dt} = 0 \neq \frac{dx}{dt}$ and vertical where $\frac{dx}{dt} = 0 \neq \frac{dy}{dt}$. Where both vanish the curve may have a cusp, and needs separate study.

COMMON ERROR. The second derivative is not $\frac{d^2y/dt^2}{d^2x/dt^2}$. Differentiate $\frac{dy}{dx}$ with respect to t , then divide by $\frac{dx}{dt}$ once more, which is the chain rule applied a second time.

Example 11.4 (A curve with a loop). Analyze $x = t^2, y = t^3 - 3t$.

Solution. $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3$, so $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$. Horizontal tangents at $t = \pm 1$, the points $(1, \mp 2)$; a vertical tangent at $t = 0$, the origin. The curve crosses itself where $t_1 \neq t_2$ give the same point: $t_1^2 = t_2^2$ forces $t_2 = -t_1$, and then $t^3 - 3t = 0$ gives $t = \pm\sqrt{3}$, the point $(3, 0)$. The two branches there have slopes $\frac{6}{\pm 2\sqrt{3}} = \pm\sqrt{3}$. Finally,

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t} = \frac{3(t^2 + 1)}{4t^3},$$

positive for $t > 0$ and negative for $t < 0$. The half traced for $t > 0$, from the origin down through $(1, -2)$ and back up through $(3, 0)$, is concave up; its mirror image, traced for $t < 0$, is concave down (Fig. 11.1).

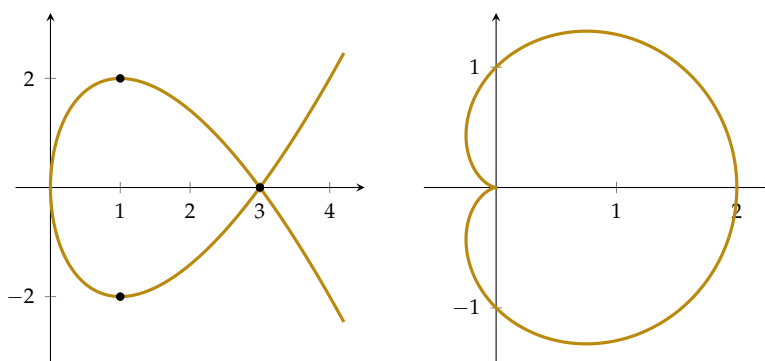


Figure 11.1: Left: the curve $x = t^2, y = t^3 - 3t$ of Example 11.4, with its horizontal tangents and its double point. Right: the cardioid $r = 1 + \cos \theta$.

11.2.2 AREAS

If the curve $x = f(t)$, $y = g(t) \geq 0$ for $\alpha \leq t \leq \beta$ traces the graph of a function from left to right exactly once, substituting $x = f(t)$ in $\int y \, dx$ gives

$$A = \int_{\alpha}^{\beta} g(t)f'(t) \, dt.$$

If the curve is traced from right to left, the integral is the negative of the area.

Example 11.5 (Under one arch of a cycloid). Find the area under one arch of the cycloid of Example 11.3.

Solution. $A = \int_0^{2\pi} r(1 - \cos \theta) \cdot r(1 - \cos \theta) \, d\theta = r^2 \int_0^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) \, d\theta = r^2(2\pi + \pi) = 3\pi r^2$, three times the area of the rolling circle, as Roberval and Torricelli found in the 1630s.

11.2.3 ARC LENGTH

Theorem 11.6. If f' and g' are continuous on $[\alpha, \beta]$ and the curve is traversed exactly once as t increases from α to β , its length is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

Proof. A partition $t_0 < \cdots < t_n$ gives an inscribed polygon whose i th side has length $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. By the Mean Value Theorem, $\Delta x_i = f'(t_i^*)\Delta t_i$ and $\Delta y_i = g'(t_i^{**})\Delta t_i$ for points t_i^*, t_i^{**} in the i th subinterval, so the polygon has length $\sum \sqrt{f'(t_i^*)^2 + g'(t_i^{**})^2} \Delta t_i$. This is nearly a Riemann sum; because g' is uniformly continuous, replacing t_i^{**} by t_i^* changes the sum by less than any ϵ once the mesh is small, and the limit is the integral. ■

For a graph $y = F(x)$, taking $t = x$ recovers the formula $\int \sqrt{1 + F'(x)^2} \, dx$ of Chapter 8. The quantity $ds = \sqrt{dx^2 + dy^2}$ is the element of arc length.

Example 11.7 (The length of a cycloid arch). **Solution.** $\frac{dx}{d\theta} = r(1 - \cos \theta)$ and $\frac{dy}{d\theta} = r \sin \theta$, so $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2(2 - 2\cos \theta) = 4r^2 \sin^2 \frac{\theta}{2}$. Then

$$L = \int_0^{2\pi} 2r \sin \frac{\theta}{2} \, d\theta = \left[-4r \cos \frac{\theta}{2} \right]_0^{2\pi} = 8r.$$

Christopher Wren found this in 1658: an arch is exactly four times the di-

ameter of the wheel.

11.2.4 SURFACE AREA

Rotating the curve about the x -axis, with $y \geq 0$, sweeps a surface of area $S = \int_{\alpha}^{\beta} 2\pi y \, ds$, by the same frustum argument as in Chapter 8; about the y -axis, $S = \int 2\pi x \, ds$ with $x \geq 0$.

Example 11.8 (A sphere). Rotating the semicircle $x = r \cos t$, $y = r \sin t$, $0 \leq t \leq \pi$, about the x -axis: $ds = r \, dt$, so $S = \int_0^{\pi} 2\pi r \sin t \cdot r \, dt = 4\pi r^2$.

11.3 POLAR COORDINATES

Choose a point O , the *pole*, and a ray from it, the *polar axis* (the positive x -axis). A point P has *polar coordinates* (r, θ) if it lies at distance $|r|$ from O along the ray at angle θ from the polar axis, when $r \geq 0$, or along the opposite ray, when $r < 0$. The conversions are

$$x = r \cos \theta, \quad y = r \sin \theta; \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

Polar coordinates are not unique: (r, θ) , $(r, \theta + 2\pi)$, and $(-r, \theta + \pi)$ are the same point, and the pole is $(0, \theta)$ for every θ . In converting to polar form, the quadrant of (x, y) decides θ ; $\arctan \frac{y}{x}$ alone is correct only for $x > 0$.

Example 11.9 (Conversions). (a) The point $(-1, \sqrt{3})$ has $r = 2$ and lies in the second quadrant, so $\theta = \frac{2\pi}{3}$. (b) The circle $x^2 + y^2 = 4y$ becomes $r^2 = 4r \sin \theta$, that is, $r = 4 \sin \theta$ (the equation $r = 0$ adds only the pole, which $r = 4 \sin \theta$ already contains). (c) The line $r = \sec \theta$ is $r \cos \theta = 1$, that is, $x = 1$.

11.3.1 POLAR CURVES

The graph of $r = f(\theta)$ is the set of points with some polar representation satisfying the equation. Standard families:

- $r = a$: a circle about the pole; $\theta = c$: a line through it.
- $r = 2a \cos \theta$ and $r = 2a \sin \theta$: circles of radius $|a|$ through the pole.

- $r = a(1 \pm \cos \theta)$: *cardioids*; $r = a + b \cos \theta$ with $a \neq b$: *limaçons*, with an inner loop when $|b| > |a|$.
- $r = a \cos n\theta$: *roses*, with n petals if n is odd and $2n$ if n is even.
- $r = a\theta$: the *spiral of Archimedes*; $r = e^{c\theta}$: the *logarithmic spiral*.

To sketch $r = f(\theta)$, graph r against θ in rectangular coordinates first, then read off how the distance changes as the ray sweeps around. Symmetry saves work: replacing θ by $-\theta$ without changing the equation means symmetry about the polar axis; replacing r by $-r$, symmetry about the pole.

11.4 CALCULUS IN POLAR COORDINATES

11.4.1 TANGENTS

The curve $r = f(\theta)$ is the parametric curve $x = f(\theta) \cos \theta$, $y = f(\theta) \sin \theta$, so

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}, \quad r' = \frac{dr}{d\theta}.$$

At the pole, $r = 0$ and this reduces to $\tan \theta$: a curve passing through the pole at angle θ_0 is tangent there to the line $\theta = \theta_0$.

Example 11.10 (Tangents to a cardioid). Find the slope of $r = 1 + \cos \theta$ at $\theta = \frac{\pi}{2}$, and the points with horizontal tangents.

Solution. $r' = -\sin \theta$. At $\theta = \frac{\pi}{2}$: $r = 1$, $r' = -1$, and $\frac{dy}{dx} = \frac{-1+0}{0-1} = 1$. In general the numerator is $-\sin^2 \theta + (1 + \cos \theta) \cos \theta = 2 \cos^2 \theta + \cos \theta - 1 = (2 \cos \theta - 1)(\cos \theta + 1)$, which vanishes for $\cos \theta = \frac{1}{2}$ and $\cos \theta = -1$. At $\theta = \pm \frac{\pi}{3}$ the denominator $-\sin \theta(1 + 2 \cos \theta) \neq 0$, giving horizontal tangents at $(\frac{3}{2}, \pm \frac{\pi}{3})$. At $\theta = \pi$ both vanish: that is the cusp at the pole, where the curve is tangent to the line $\theta = \pi$, the x -axis.

11.4.2 AREA

Theorem 11.11. If f is continuous and nonnegative on $[\alpha, \beta]$, with $0 < \beta - \alpha \leq 2\pi$, the region bounded by $r = f(\theta)$ and the rays $\theta = \alpha$, $\theta = \beta$ has area

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta.$$

Proof. Partition $[\alpha, \beta]$. The part of the region between θ_{i-1} and θ_i contains the circular sector of radius $m_i = \min f$ on that interval and lies in the sector of radius $M_i = \max f$. A sector of radius ρ and angle $\Delta\theta$ has area $\frac{1}{2}\rho^2\Delta\theta$. Hence the area lies between the lower and upper sums $\sum \frac{1}{2}m_i^2\Delta\theta_i$ and $\sum \frac{1}{2}M_i^2\Delta\theta_i$ for $\frac{1}{2}f^2$, which converge to the integral. ■

Example 11.12 (Cardioid and rose). The cardioid $r = 1 + \cos \theta$ encloses $\int_0^{2\pi} \frac{1}{2}(1 + \cos \theta)^2 d\theta = \frac{1}{2}(2\pi + \pi) = \frac{3\pi}{2}$. The four-petaled rose $r = \cos 2\theta$ encloses $\int_0^{2\pi} \frac{1}{2} \cos^2 2\theta d\theta = \frac{\pi}{2}$, where integrating over $[0, 2\pi]$ traces each petal exactly once. (Integrating $\frac{1}{2} \cos^2 3\theta$ over $[0, 2\pi]$ for the three-petaled rose $r = \cos 3\theta$ would count every petal twice, since that rose is traced completely on $[0, \pi]$.)

Example 11.13 (Between two polar curves). Find the area inside $r = 3 \sin \theta$ and outside $r = 1 + \sin \theta$.

Solution. The curves meet where $3 \sin \theta = 1 + \sin \theta$, so $\sin \theta = \frac{1}{2}$: $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Between these rays the circle is outside the cardioid, and

$$\begin{aligned} A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2} (9 \sin^2 \theta - (1 + \sin \theta)^2) d\theta \\ &= \int_{\pi/6}^{5\pi/6} (4 \sin^2 \theta - \sin \theta - \frac{1}{2}) d\theta = \pi. \end{aligned}$$

Both curves also pass through the pole, but at different values of θ : solving the equations simultaneously finds only intersections reached at the same θ , so a sketch is needed to find them all.

11.4.3 ARC LENGTH

As a parametric curve in θ , $(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (r' \cos \theta - r \sin \theta)^2 + (r' \sin \theta + r \cos \theta)^2 = r^2 + r'^2$, so

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

For the cardioid $r = 1 + \cos \theta$, $r^2 + r'^2 = 2 + 2 \cos \theta = 4 \cos^2 \frac{\theta}{2}$, and $L = 2 \int_0^{\pi} 2 \cos \frac{\theta}{2} d\theta = 8$.

11.5 CONIC SECTIONS

The parabola, ellipse, and hyperbola, studied by Apollonius as sections of a cone, have a single unifying description.

Definition 11.14. Let F be a point (the *focus*), ℓ a line not through F (the *directrix*), and $e > 0$ (the *eccentricity*). The set of points P with $\frac{|PF|}{\text{dist}(P, \ell)} = e$ is a *conic section*: an *ellipse* if $e < 1$, a *parabola* if $e = 1$, and a *hyperbola* if $e > 1$.

Theorem 11.15. With the focus at the pole and the directrix $x = d > 0$, the conic of eccentricity e has polar equation

$$r = \frac{ed}{1 + e \cos \theta}.$$

Proof. For $P = (r, \theta)$ on the focus side of the directrix, $|PF| = r$ and $\text{dist}(P, \ell) = d - r \cos \theta$. The condition $r = e(d - r \cos \theta)$ rearranges to $r(1 + e \cos \theta) = ed$. ■

For $e < 1$ the denominator is positive for all θ , and the curve is closed: the nearest point is at $\theta = 0$, distance $\frac{ed}{1+e}$; the farthest at $\theta = \pi$, distance $\frac{ed}{1-e}$. The *major axis* has length $2a = \frac{ed}{1+e} + \frac{ed}{1-e} = \frac{2ed}{1-e^2}$, so $ed = a(1 - e^2)$ and

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}, \quad r_{\min} = a(1 - e), \quad r_{\max} = a(1 + e).$$

The distance from center to focus is $c = ae$, and $b^2 = a^2 - c^2$ recovers the rectangular form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the center. As $e \rightarrow 0$, the ellipse becomes a circle; as $e \rightarrow 1$ with the nearest distance fixed, it lengthens toward a parabola.

Example 11.16 (Halley's comet). Halley's comet moves on an ellipse with the Sun at a focus, eccentricity $e \approx 0.967$, and perihelion (closest) distance 0.586 AU. Find its aphelion distance and period.

Solution. $a = \frac{0.586}{1-0.967} \approx 17.8$ AU, and the aphelion is $a(1 + e) \approx 34.9$ AU, beyond the orbit of Neptune. By Kepler's third law, the period in years is $a^{3/2} \approx 74.8$, with a in AU; the observed period, which varies with planetary perturbations, is 74 to 79 years.

LOOKING AHEAD. Kepler's laws state that planets move on ellipses with the Sun at a focus, sweeping out equal areas in equal times. In polar form,

the area swept is $\int \frac{1}{2} r^2 d\theta$, so the second law says $r^2 \frac{d\theta}{dt}$ is constant: the conservation of angular momentum. In Volume IV, Newton's derivation of the ellipse from the inverse-square law is carried out with vector calculus, and the same $\frac{1}{1+e \cos \theta}$ reappears in the classical picture of the hydrogen atom.

11.6 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 11.1 The path of a projectile

A ball is thrown from the ground at 20 m/s at angle 45° . Ignoring air resistance ($g = 9.8 \text{ m/s}^2$), find its path, range, maximum height, and the distance it travels through the air.

Solution. Integrating the acceleration $(0, -g)$ twice, with initial velocity $(v \cos \alpha, v \sin \alpha)$,

$$x = (v \cos \alpha)t, \quad y = (v \sin \alpha)t - \frac{1}{2}gt^2.$$

Eliminating t gives the parabola $y = x \tan \alpha - \frac{g}{2v^2 \cos^2 \alpha} x^2$. The ball lands when $y = 0$, at $t = \frac{2v \sin \alpha}{g}$, so the range is $\frac{v^2 \sin 2\alpha}{g}$, largest at $\alpha = 45^\circ$: $\frac{400}{9.8} \approx 40.8$ m. The height peaks where $\frac{dy}{dt} = 0$, at $\frac{(v \sin \alpha)^2}{2g} \approx 10.2$ m. The distance traveled is

$$L = \int_0^{2.886} \sqrt{(14.142)^2 + (14.142 - 9.8t)^2} dt \approx 46.8 \text{ m},$$

which can be done exactly with $\int \sqrt{a^2 + u^2} du$ from Chapter 9, but is quicker numerically. The path is about 15% longer than the range.

— Problem 11.2 The folium of Descartes

Find the area of the loop of $x^3 + y^3 = 3axy$, $a > 0$.

Solution. In polar coordinates, $r^3(\cos^3 \theta + \sin^3 \theta) = 3ar^2 \cos \theta \sin \theta$, so $r = \frac{3a \sin \theta \cos \theta}{\sin^3 \theta + \cos^3 \theta}$. The loop is traced for $0 \leq \theta \leq \frac{\pi}{2}$ (where $r \geq 0$ and the loop returns to the pole). Hence

$$A = \frac{1}{2} \int_0^{\pi/2} \frac{9a^2 \sin^2 \theta \cos^2 \theta}{(\sin^3 \theta + \cos^3 \theta)^2} d\theta.$$

Divide numerator and denominator by $\cos^6 \theta$: the integrand becomes $\frac{9a^2 \tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2}$. With $u = 1 + \tan^3 \theta$, $du = 3 \tan^2 \theta \sec^2 \theta d\theta$, and u runs from 1 to ∞ :

$$A = \frac{1}{2} \int_1^\infty \frac{3a^2}{u^2} du = \frac{3a^2}{2}.$$

The polar form turned an implicit cubic, which is hard to integrate in x and y , into a substitution exercise.

— Problem 11.3 The equiangular spiral

Show that the logarithmic spiral $r = e^\theta$ crosses every ray from the pole at the same angle, and find the length of the part with $\theta \leq 0$.

Solution. Let ψ be the angle between the ray (direction θ) and the tangent. The tangent direction is that of $(\frac{dx}{d\theta}, \frac{dy}{d\theta})$, whose components along and perpendicular to the ray are r' and r (expand $(r' \cos \theta - r \sin \theta, r' \sin \theta + r \cos \theta)$ in the basis $(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)$). So $\tan \psi = \frac{r}{r'}$. For $r = e^\theta$, $r' = r$ and $\psi = \frac{\pi}{4}$ everywhere. The part with $\theta \leq 0$ winds infinitely often around the pole, yet its length is finite:

$$L = \int_{-\infty}^0 \sqrt{e^{2\theta} + e^{2\theta}} d\theta = \sqrt{2} \int_{-\infty}^0 e^\theta d\theta = \sqrt{2}.$$

Jakob Bernoulli, who called it the *spira mirabilis*, asked for it to be carved on his tombstone. The mason carved an Archimedean spiral instead.

— Problem 11.4 Two cardioids

Find the area common to $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

Solution. The cardioids meet where $\cos \theta = 0$, and at the pole. By symmetry about both axes, the common region is four copies of its part in the first quadrant, where the smaller curve is $r = 1 - \cos \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$:

$$A = 4 \int_0^{\pi/2} \frac{1}{2} (1 - \cos \theta)^2 d\theta = 2 \left(\frac{\pi}{2} - 2 + \frac{\pi}{4} \right) = \frac{3\pi}{2} - 4 \approx 0.712.$$

Each cardioid has area $\frac{3\pi}{2}$, so they share only about 15% of their area.

— Problem 11.5 The ellipse from its focus

Show that for $0 < e < 1$ the conic $r = \frac{ed}{1 + e \cos \theta}$ is the ellipse $\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a = \frac{ed}{1-e^2}$, $c = ae$, $b^2 = a^2 - c^2$, and that the sum of the distances to the two foci is $2a$.

Solution. From $r = e(d - x)$, square: $x^2 + y^2 = e^2(d - x)^2$, so $(1 - e^2)x^2 + 2e^2dx + y^2 = e^2d^2$. Complete the square: $(1 - e^2)(x + \frac{e^2d}{1 - e^2})^2 + y^2 = e^2d^2 + \frac{e^4d^2}{1 - e^2} = \frac{e^2d^2}{1 - e^2}$. Dividing by $\frac{e^2d^2}{1 - e^2}$,

$$\frac{(x + c)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1, \quad a = \frac{ed}{1 - e^2}, \quad c = \frac{e^2d}{1 - e^2} = ae.$$

So $b^2 = a^2(1 - e^2) = a^2 - c^2$, and the center is at $(-c, 0)$, with foci at the pole and at $(-2c, 0)$. For the focal sum, the distance to the first focus is $r = a - ex$ (since $r = ed - ex = a(1 - e^2) - ex$ and, with $X = x + c$ measured from the center, $r = a - eX$); by symmetry the distance to the other is $a + eX$. Their sum is $2a$: the gardener's definition of the ellipse, with a loop of string around two pegs.

— Problem 11.6 A torus

Find the surface area of the torus formed by rotating the circle $(x - R)^2 + y^2 = a^2$, $0 < a < R$, about the y -axis.

Solution. Parametrize: $x = R + a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$, with $ds = a dt$. About the y -axis,

$$S = \int_0^{2\pi} 2\pi(R + a \cos t) a dt = 2\pi a(2\pi R) = 4\pi^2 Ra.$$

This is *Pappus's theorem*: the area of a surface of revolution is the length of the generating curve ($2\pi a$) times the distance traveled by its centroid ($2\pi R$). The volume of the torus, $2\pi^2 Ra^2$, follows the same way from the area of the disk.

EXERCISES

A. COMPUTATION

11.1 Eliminate the parameter and sketch: $x = 3t - 1$, $y = 2t + 5$.

11.2 Eliminate the parameter and sketch: $x = e^t$, $y = e^{-2t}$.

11.3 Eliminate the parameter, and identify the part of the curve traced: $x = \sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

11.4 Eliminate the parameter: $x = \sin t$, $y = \cos 2t$. What part of the parabola is traced, and how often, for $0 \leq t \leq 2\pi$?

11.5 Parametrize the segment from $(1, 2)$ to $(4, -3)$.

- 11.6** Parametrize the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ clockwise, starting at $(0, 2)$.
- 11.7** Find the tangent line to $x = t^3 + 1, y = t^2 - t$ at $t = 2$.
- 11.8** Find the tangent line to $x = 2 \cos t, y = 3 \sin t$ at $t = \frac{\pi}{4}$.
- 11.9** Find the points of $x = t^3 - 3t, y = t^2 - 4$ with horizontal or vertical tangents.
- 11.10** Find $\frac{d^2y}{dx^2}$ for $x = e^t, y = te^{-t}$.
- 11.11** Find $\frac{d^2y}{dx^2}$ for the cycloid $x = t - \sin t, y = 1 - \cos t$, and show that it is concave down on every arch.
- 11.12** Find the area enclosed by $x = a \cos t, y = b \sin t$.
- 11.13** Find the area under $x = t^2, y = e^{-t}, 0 \leq t \leq 1$.
- 11.14** Find the area enclosed by the astroid $x = \cos^3 t, y = \sin^3 t$.
- 11.15** Find the length of $x = t^2, y = t^3, 0 \leq t \leq 1$.
- 11.16** Find the length of $x = e^t \cos t, y = e^t \sin t, 0 \leq t \leq \pi$.
- 11.17** Find the length of the nephroid arc $x = 3 \cos t - \cos 3t, y = 3 \sin t - \sin 3t, 0 \leq t \leq \pi$.
- 11.18** Find the area of the surface formed by rotating the astroid $x = a \cos^3 t, y = a \sin^3 t$ about the x -axis.
- 11.19** Find the area of the surface formed by rotating $x = 3t^2, y = 2t^3, 0 \leq t \leq 1$, about the y -axis.
- 11.20** Convert to rectangular coordinates: $(2, \frac{7\pi}{4}), (-3, \frac{\pi}{6}), (4, -\frac{2\pi}{3})$.
- 11.21** Convert to polar coordinates with $r > 0, 0 \leq \theta < 2\pi$: $(-3, 3), (1, -\sqrt{3})$.
- 11.22** Convert to rectangular form and identify: $r = 6 \cos \theta, r = \frac{2}{\sin \theta + \cos \theta}, r^2 \sin 2\theta = 4$.
- 11.23** Convert to polar form: $x^2 - y^2 = 1, y = x^2$.
- 11.24** Sketch $r = 1 - 2 \cos \theta$, and find the values of θ at which it passes through the pole.
- 11.25** Find the slope of the spiral $r = \theta$ at $\theta = \frac{\pi}{2}$.
- 11.26** Find the points on $r = 2 \sin \theta$ with horizontal and vertical tangents.

- 11.27** Find the area of one petal of $r = \sin 3\theta$.
- 11.28** Find the area inside $r = 2 \cos \theta$ and outside $r = 1$.
- 11.29** Find the area of the inner loop of $r = 1 + 2 \cos \theta$.
- 11.30** Find the length of $r = \theta^2$, $0 \leq \theta \leq 2\pi$, to three decimal places.
- 11.31** Identify the conic $r = \frac{6}{2 + \cos \theta}$, and find its eccentricity, nearest and farthest distances from the pole, and semi-major axis.
- 11.32** Find the polar equation of the parabola with focus at the pole and directrix $y = -3$.

B. THEORY AND PROOF

- 11.33** Prove that the tangent to the cycloid at any point passes through the top of the rolling circle.
- 11.34** Show that the polar curve $r = f(\theta)$ through the pole at $\theta = \theta_0$ (with $f'(\theta_0) \neq 0$) is tangent to the line $\theta = \theta_0$ there.
- 11.35** Show that the area of a polar region is invariant under rotation: $r = f(\theta - \phi)$ encloses the same area as $r = f(\theta)$.
- 11.36** Show that for the conic $r = \frac{ed}{1 + e \cos \theta}$ with $e > 1$, the denominator vanishes at two angles, and that these are the directions of the asymptotes of the hyperbola.
- 11.37** Prove Kepler's second law in this form: if $r^2 \frac{d\theta}{dt} = h$ is constant, then the area swept out from time t_1 to t_2 is $\frac{h}{2}(t_2 - t_1)$.
- 11.38** Prove the reflection property of the parabola $y^2 = 4px$: the tangent at any point makes equal angles with the line to the focus $(p, 0)$ and the horizontal.

C. CHALLENGE

- 11.39** Show that the time for a bead to slide without friction down an inverted cycloid $x = r(\theta - \sin \theta)$, $y = -r(1 - \cos \theta)$ from $\theta = \theta_0$ to the bottom ($\theta = \pi$) is $\pi\sqrt{r/g}$, independent of θ_0 : the tautochrone property.
- 11.40** The *lemniscate* $r^2 = \cos 2\theta$. Find its area, and show that its total length is $4 \int_0^1 \frac{dt}{\sqrt{1-t^4}}$, an elliptic integral studied by Fagnano and Euler.

- 11.41** A dog at $(1, 0)$ runs toward its owner, who walks up the y -axis from the origin at the same speed; the dog always heads directly at the owner. Find the dog's path (the *pursuit curve*) and show that the dog never catches up.
- 11.42** Find the ratio of the length of an ellipse with $e = 0.5$ to the circumference of the circle with the same semi-major axis, to four decimal places.
- 11.43** The *epicycloid* traced by a point on a circle of radius b rolling outside a fixed circle of radius a , with $\frac{a}{b} = n$ an integer, has n cusps. Find its length and enclosed area.

CHAPTER 12

INFINITE SEQUENCES AND SERIES

LEARNING OBJECTIVES

- Determine the limits of sequences, including recursive sequences, using the Monotone Convergence Theorem.
- Sum geometric and telescoping series, and apply the divergence test.
- Apply the integral, comparison, and limit comparison tests, and bound the remainder of a series.
- Distinguish absolute from conditional convergence, and apply the alternating series test with its error bound.
- Apply the ratio and root tests, and choose an efficient test for a given series.

What does it mean to add infinitely many numbers? Zeno argued that a runner can never finish a race, since she must first cover half the distance, then half the remainder, and so on forever. The answer, that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$, requires a precise definition of an infinite sum, and that definition, as a limit of finite sums, is the subject of this chapter. Its payoff comes in Chapter 13, where functions such as e^x and $\sin x$ are written as infinite sums of powers of x .

12.1 SEQUENCES

Definition 12.1. A *sequence* is a function on the positive integers, written (a_n) or $a_1, a_2, a_3, \dots$. It *converges* to L , written $\lim_{n \rightarrow \infty} a_n = L$, if for every $\epsilon > 0$ there is an N such that $|a_n - L| < \epsilon$ for all $n > N$. A sequence that does not converge *diverges*; it diverges to ∞ if for every M , $a_n > M$ for all large n .

The limit laws, the squeeze theorem, and continuity carry over from functions

(Chapter 2) with the same proofs: if $a_n \rightarrow L$ and f is continuous at L , then $f(a_n) \rightarrow f(L)$. If $a_n = f(n)$ for a function with $\lim_{x \rightarrow \infty} f(x) = L$, then $a_n \rightarrow L$, so L'Hôpital's rule can be applied to the function. The converse fails: $\sin(\pi n) = 0$ for every integer n , but $\sin \pi x$ has no limit.

Example 12.2 (Standard limits). (a) $\frac{\ln n}{n} \rightarrow 0$, by L'Hôpital's rule on $\frac{\ln x}{x}$. (b) $n^{1/n} = e^{(\ln n)/n} \rightarrow e^0 = 1$. (c) $r^n \rightarrow 0$ if $|r| < 1$, $r^n \rightarrow \infty$ if $r > 1$, and $(-1)^n$ diverges. (d) $(1 + \frac{x}{n})^n \rightarrow e^x$ (Chapter 6). (e) $\frac{x^n}{n!} \rightarrow 0$ for every x : once $n > 2|x|$, each further factor $\frac{|x|}{n}$ is less than $\frac{1}{2}$.

Definition 12.3. A sequence is *increasing* if $a_n \leq a_{n+1}$ for all n , *decreasing* if $a_n \geq a_{n+1}$, and *monotone* if either. It is *bounded* if $|a_n| \leq M$ for some M and all n .

Theorem 12.4 (Monotone Convergence Theorem). *Every bounded monotone sequence converges.*

Proof. Let (a_n) be increasing and bounded above. By the completeness of the real numbers (Appendix A), the set $\{a_n\}$ has a least upper bound L . Given $\epsilon > 0$, $L - \epsilon$ is not an upper bound, so $a_N > L - \epsilon$ for some N . For $n > N$, $L - \epsilon < a_N \leq a_n \leq L$, so $|a_n - L| < \epsilon$. The decreasing case is the same with greatest lower bounds. ■

The theorem guarantees a limit without naming it. For recursive sequences, the limit can then be found by passing to the limit in the recursion.

Example 12.5 (A nested radical). Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Show that (a_n) converges and find its limit.

Solution. By induction, $a_n < 2$: true for $n = 1$, and if $a_n < 2$ then $a_{n+1} < \sqrt{4} = 2$. And a_n is increasing: $a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2 - a_n)(1 + a_n) > 0$. So the sequence converges to some L , and letting $n \rightarrow \infty$ in $a_{n+1}^2 = 2 + a_n$ gives $L^2 = 2 + L$, so $L = 2$ (the root -1 is excluded since $a_n > 0$). The terms 1.414, 1.848, 1.962, 1.990, 1.998, ... confirm it.

COMMON ERROR. Passing to the limit in a recursion is valid only after convergence is known. The recursion $a_{n+1} = 2a_n$, $a_1 = 1$, gives " $L = 2L$, so

$L = 0''$, although $a_n = 2^{n-1} \rightarrow \infty$.

12.2 INFINITE SERIES

Definition 12.6. The series $\sum_{n=1}^{\infty} a_n$ has *partial sums* $s_N = a_1 + \cdots + a_N$. If $s_N \rightarrow s$, the series *converges* with *sum* s ; otherwise it *diverges*.

Theorem 12.7 (Geometric series). For $a \neq 0$, the series $\sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$, and then its sum is $\frac{a}{1-r}$.

Proof. For $r \neq 1$, $s_N = a + ar + \cdots + ar^{N-1} = a \frac{1-r^N}{1-r}$ (multiply by $1-r$ and telescope). If $|r| < 1$, $r^N \rightarrow 0$ and $s_N \rightarrow \frac{a}{1-r}$. If $|r| \geq 1$, the terms ar^n do not tend to 0, so the series diverges by the next theorem. ■

Example 12.8 (Repeating decimals). $0.\overline{27} = \frac{27}{100} + \frac{27}{100^2} + \cdots = \frac{27/100}{1-1/100} = \frac{27}{99} = \frac{3}{11}$. Every repeating decimal is a geometric series and so a rational number; in particular $0.\overline{9} = \frac{9/10}{1-1/10} = 1$.

Example 12.9 (Telescoping series). $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum \left(\frac{1}{n} - \frac{1}{n+1} \right)$ has $s_N = 1 - \frac{1}{N+1} \rightarrow 1$. Similarly, $\frac{1}{n^2-1} = \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$, and the sum from $n = 2$ telescopes to $\frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}$: here each term cancels with the one two places later, leaving two terms at the start.

Theorem 12.10 (Divergence test). If $\sum a_n$ converges, then $a_n \rightarrow 0$. Equivalently, if $a_n \not\rightarrow 0$, the series diverges.

Proof. $a_n = s_n - s_{n-1} \rightarrow s - s = 0$. ■

The converse is false, and the most important counterexample is the *harmonic series*.

Theorem 12.11. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. Group the terms in blocks ending at powers of 2: $\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$, $\frac{1}{5} + \cdots + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$, and in general the 2^{k-1} terms from $\frac{1}{2^{k-1}+1}$ to $\frac{1}{2^k}$ exceed $2^{k-1} \cdot \frac{1}{2^k} = \frac{1}{2}$. So $s_{2^k} \geq 1 + \frac{k}{2} \rightarrow \infty$. (Oresme, around 1350.) ■

The divergence is slow: $s_N - \ln N$ tends to *Euler's constant* $\gamma \approx 0.5772$, so a million terms sum to only about 14.4.

Convergent series can be added and multiplied by constants termwise, and changing finitely many terms changes the sum but not whether the series converges.

12.3 THE INTEGRAL TEST

Theorem 12.12 (Integral test). *Let f be continuous, positive, and decreasing on $[1, \infty)$, with $a_n = f(n)$. Then $\sum a_n$ and $\int_1^\infty f(x) dx$ converge or diverge together. If they converge, the remainder $R_N = s - s_N$ satisfies*

$$\int_{N+1}^\infty f(x) dx \leq R_N \leq \int_N^\infty f(x) dx.$$

Proof. Since f is decreasing, $a_{n+1} \leq \int_n^{n+1} f \leq a_n$. Summing the left inequality from $n = 1$ to $N - 1$ gives $s_N - a_1 \leq \int_1^N f$; summing the right gives $\int_1^{N+1} f \leq s_N$. The partial sums are increasing, so by Theorem 12.4 they converge if and only if they are bounded, which by these inequalities happens if and only if the integral converges. Summing the same inequalities from $n = N$ to ∞ bounds $R_N = \sum_{n>N} a_n$. ■

Corollary 12.13 (p -series). $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. For $p > 0$ apply the integral test with $f(x) = x^{-p}$; $\int_1^\infty x^{-p} dx$ converges exactly when $p > 1$ (Chapter 9). For $p \leq 0$ the terms do not tend to 0. ■

Example 12.14 (Estimating $\zeta(3)$). Estimate $\sum \frac{1}{n^3}$ using ten terms, with a guaranteed error bound.

Solution. $s_{10} \approx 1.197532$. With $\int_N^\infty x^{-3} dx = \frac{1}{2N^2}$, the remainder lies be-

tween $\frac{1}{2.11^2} \approx 0.004132$ and $\frac{1}{200} = 0.005$, so

$$1.201664 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.202532.$$

The midpoint 1.202098 is within 0.00005 of the true value 1.2020569, *Apéry's constant*, proved irrational only in 1978. By contrast, Euler showed in 1735 that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (Chapter 13).

12.4 COMPARISON TESTS

Theorem 12.15 (Comparison test). Suppose $0 \leq a_n \leq b_n$ for all n . If $\sum b_n$ converges, so does $\sum a_n$; if $\sum a_n$ diverges, so does $\sum b_n$.

Proof. The partial sums of $\sum a_n$ are increasing and bounded by $\sum b_n$, so they converge by Theorem 12.4. The second statement is the contrapositive of the first. ■

Theorem 12.16 (Limit comparison test). Suppose $a_n, b_n > 0$ and $\frac{a_n}{b_n} \rightarrow c$ with $0 < c < \infty$. Then $\sum a_n$ and $\sum b_n$ converge or diverge together.

Proof. For large n , $\frac{c}{2} < \frac{a_n}{b_n} < 2c$, so $a_n < 2cb_n$ and $b_n < \frac{2}{c}a_n$. Apply the comparison test both ways. ■

The limit comparison test makes precise the idea that only the dominant behavior of the terms matters: compare with the p -series or geometric series obtained by keeping the leading powers.

Example 12.17 (Comparisons). (a) $\sum \frac{1}{n^2+n+5}$: $\frac{1}{n^2+n+5} < \frac{1}{n^2}$, so it converges. (b) $\sum \frac{n+3}{\sqrt{n^5+1}}$: the terms behave like $\frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$, and the limit of the ratio is 1, so it converges. (c) $\sum \frac{1}{2n-1}$: comparing with $\frac{1}{n}$ gives ratio $\frac{1}{2}$, so it diverges. (d) $\sum \frac{\ln n}{n^2}$: since $\ln n < \sqrt{n}$ for all n , the terms are less than $\frac{1}{n^{3/2}}$, and it converges.

COMMON ERROR. The comparison must go the right way. Showing that $a_n \leq b_n$ with $\sum b_n$ divergent proves nothing about $\sum a_n$: $\frac{1}{n^2} \leq \frac{1}{n}$, yet $\sum \frac{1}{n^2}$ converges.

12.5 ALTERNATING SERIES AND ABSOLUTE CONVERGENCE

Theorem 12.18 (Alternating series test). *If b_n is decreasing with $b_n \rightarrow 0$, then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges, and its sum s satisfies $|s - s_N| \leq b_{N+1}$: the error is at most the first omitted term.*

Proof. The even partial sums $s_{2k} = (b_1 - b_2) + \cdots + (b_{2k-1} - b_{2k})$ increase, since each bracket is nonnegative. They are bounded by b_1 , since $s_{2k} = b_1 - (b_2 - b_3) - \cdots - b_{2k}$. So $s_{2k} \rightarrow s$ by Theorem 12.4, and $s_{2k+1} = s_{2k} + b_{2k+1} \rightarrow s$ too. The odd partial sums decrease toward s and the even ones increase toward it, so s lies between any two consecutive partial sums, which differ by b_{N+1} . ■

Example 12.19 (The alternating harmonic series). $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges, and in Chapter 13 its sum is shown to be $\ln 2 \approx 0.693147$. The convergence is slow: $s_{10} \approx 0.645635$, and guaranteeing an error below 0.001 requires 1000 terms. In contrast, $\sum \frac{(-1)^n}{n!} = \frac{1}{e}$, and eight terms ($n = 0$ to 7) give 0.367857, with error at most $\frac{1}{8!} \approx 0.000025$.

Definition 12.20. $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. A convergent series that does not converge absolutely converges conditionally.

Theorem 12.21. *An absolutely convergent series converges.*

Proof. $0 \leq a_n + |a_n| \leq 2|a_n|$, so $\sum (a_n + |a_n|)$ converges by comparison, and $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ is a difference of convergent series. ■

The alternating harmonic series converges conditionally. The distinction matters because sums of conditionally convergent series depend on the order of the terms.

Theorem 12.22 (Riemann rearrangement theorem). *Every rearrangement of an absolutely convergent series converges to the same sum. If $\sum a_n$ converges conditionally, then for every real S there is a rearrangement whose sum is S .*

Sketch. For the second statement, the positive terms and the negative terms each form a divergent series (if either converged, so would the other, and the series would converge absolutely), while the terms tend to 0. Add positive terms until the partial sum exceeds S , then negative terms until it falls below S , and repeat. Each overshoot is at most the size of the last term used, which tends to 0. ■

Example 12.23 (Rearranging $\ln 2$). Write the alternating harmonic series with one positive term followed by two negative terms:

$$\begin{aligned} \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots \\ = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots = \frac{1}{2} \ln 2. \end{aligned}$$

The same terms, in a different order, have half the sum.

12.6 THE RATIO AND ROOT TESTS

Theorem 12.24 (Ratio test). *Suppose $\left|\frac{a_{n+1}}{a_n}\right| \rightarrow L$. If $L < 1$, $\sum a_n$ converges absolutely; if $L > 1$ (including $L = \infty$), it diverges; if $L = 1$ the test gives no information.*

Proof. If $L < 1$, choose r with $L < r < 1$. For $n \geq N$, $|a_{n+1}| \leq r|a_n|$, so $|a_{N+k}| \leq |a_N|r^k$, and $\sum |a_n|$ converges by comparison with a geometric series. If $L > 1$, $|a_n|$ eventually increases, so $a_n \not\rightarrow 0$. For $L = 1$: both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ have ratio limit 1. ■

Theorem 12.25 (Root test). *Suppose $|a_n|^{1/n} \rightarrow L$. If $L < 1$, $\sum a_n$ converges absolutely; if $L > 1$, it diverges; if $L = 1$ the test is inconclusive.*

Proof. If $L < r < 1$, then $|a_n| < r^n$ for large n ; compare with $\sum r^n$. If $L > 1$, $|a_n| > 1$ for large n . ■

Example 12.26 (Ratio and root tests). (a) $\sum \frac{n^2}{2^n}$: $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2} \rightarrow \frac{1}{2}$, so it converges; its sum is 6 (Chapter 13). (b) $\sum \frac{n!}{n^n}$: $\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \left(\frac{n}{n+1}\right)^n \rightarrow \frac{1}{e} < 1$, so it converges. (c) $\sum \left(\frac{n}{2n+1}\right)^n$: the n th root is $\frac{n}{2n+1} \rightarrow \frac{1}{2}$, so it converges. (d) $\sum \frac{3^n}{n^3}$: ratio $\rightarrow 3$, so it diverges.

12.6.1 CHOOSING A TEST

1. If $a_n \not\rightarrow 0$, the series diverges. Check this first.
2. Geometric, p -series, and telescoping series can be recognized and decided directly.
3. For positive terms that are algebraic in n (ratios of powers and roots), use limit comparison with a p -series.
4. For factorials and exponentials, use the ratio test; for n th powers, the root test.
5. For alternating terms, test absolute convergence first; if that fails, use the alternating series test.
6. If $a_n = f(n)$ with f easy to integrate, the integral test applies, and also gives error bounds.

LOOKING AHEAD. Convergence of series is the foundation for representing functions by infinite sums. Power series, in the next chapter, are the simplest case. Fourier series, sums of sines and cosines, are treated in Volume IV, where the wave function of a particle in a box is written as such a series, and absolute convergence becomes the condition $\sum |c_n|^2 < \infty$ that defines the space of quantum states.

12.7 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 12.1 A bouncing ball

A ball dropped from 10 m rebounds each time to $\frac{3}{4}$ of its previous height. Find

the total distance it travels and the total time before it comes to rest ($g = 9.8 \text{ m/s}^2$).

Solution. After the initial fall of 10 m, the ball rises and falls $10\left(\frac{3}{4}\right)^k$ on the k th bounce, so the distance is

$$10 + 2 \sum_{k=1}^{\infty} 10\left(\frac{3}{4}\right)^k = 10 + 2 \cdot \frac{7.5}{1 - 0.75} = 70 \text{ m.}$$

A fall from height h takes $\sqrt{2h/g}$, so the k th bounce takes $2\sqrt{2h_k/g} = 2\sqrt{20/g}(\sqrt{3/4})^k$. The times also form a geometric series, with ratio $\sqrt{3/4} \approx 0.866$:

$$T = \sqrt{\frac{20}{g}} \left(1 + \frac{2\sqrt{3/4}}{1 - \sqrt{3/4}}\right) \approx 19.9 \text{ s.}$$

Infinitely many bounces happen in finite time, Zeno's paradox made physical. (A real ball stops sooner, because the model of a constant restitution ratio fails at small heights.)

— Problem 12.2 Series at the edge of convergence

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge.

Solution. Both terms are positive and decreasing, so the integral test applies. With $u = \ln x$,

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \infty, \quad \int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \frac{1}{\ln 2}.$$

The first diverges and the second converges. Both lie between the harmonic series and every p -series with $p > 1$, so the p -series do not mark a sharp boundary: there is no “slowest divergent series”. The first diverges very slowly; its partial sums grow like $\ln \ln N$, which at $N = 10^{100}$ is only about 5.4.

— Problem 12.3 A sum of Fibonacci numbers

The Fibonacci numbers are $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. Show that $\sum_{n=1}^{\infty} \frac{F_n}{2^n}$ converges and find its sum.

Solution. Since $F_{n+1} \leq 2F_n$, $F_n \leq 2^{n-1}$, which is not enough: the terms would be bounded by $\frac{1}{2}$. Better, $\frac{F_{n+1}}{F_n}$ tends to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ (Exercise 12.33), so the ratio test gives $\frac{\varphi}{2} < 1$ and the series converges, to S say. Then use the recursion:

$$S = \frac{1}{2} + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{F_{n-1} + F_{n-2}}{2^n} = \frac{3}{4} + \frac{1}{2} \left(S - \frac{1}{2}\right) + \frac{1}{4}S.$$

So $S = \frac{3}{4} + \frac{1}{2}S - \frac{1}{4} + \frac{1}{4}S$, giving $\frac{1}{4}S = \frac{1}{2}$, and $S = 2$. The shifts were legitimate only because S was already known to be finite, the same lesson as the pitfall on recursive limits.

— Problem 12.4 Deleting the nines

Show that the sum of $\frac{1}{n}$ over the positive integers n whose decimal digits contain no 9 converges, with sum less than 80.

Solution. Count the k -digit integers with no 9: 8 choices for the leading digit (not 0 or 9) and 9 for each other, so $8 \cdot 9^{k-1}$. Each is at least 10^{k-1} , so their reciprocals sum to at most $\frac{8 \cdot 9^{k-1}}{10^{k-1}} = 8\left(\frac{9}{10}\right)^{k-1}$. Summing over k ,

$$\sum \frac{1}{n} < \sum_{k=1}^{\infty} 8\left(\frac{9}{10}\right)^{k-1} = \frac{8}{1-0.9} = 80.$$

This is Kempner's series (1914); its sum is about 22.92. The paradox dissolves on counting: almost all large integers contain a 9, and the harmonic series diverges only because of them.

— Problem 12.5 Accuracy of an alternating sum

How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ guarantee an error below 10^{-6} ? Is the sum larger or smaller than s_N for that N ?

Solution. By Theorem 12.18 the error after N terms is at most $\frac{1}{(N+1)^4}$. Since $31^4 = 923\,521 < 10^6 < 32^4 = 1\,048\,576$, we need $N+1 = 32$, so 31 terms. The partial sum s_{31} ends with a positive term, and the proof of the theorem shows that the odd partial sums lie above the sum: $s < s_{31}$. (The exact sum is $\frac{7\pi^4}{720}$.)

— Problem 12.6 A series with a parameter

For which x does $\sum_{n=1}^{\infty} \frac{nx^n}{3^n}$ converge?

Solution. Ratio test: $\left| \frac{(n+1)x^{n+1}/3^{n+1}}{nx^n/3^n} \right| = \frac{n+1}{n} \cdot \frac{|x|}{3} \rightarrow \frac{|x|}{3}$. The series converges absolutely for $|x| < 3$ and diverges for $|x| > 3$. At $x = \pm 3$ the ratio test is silent, but the terms are n and $(-1)^n n$, which do not tend to 0: divergence. The set of convergence is the open interval $(-3, 3)$. Every series of the form $\sum c_n x^n$ converges on an interval like this, as Chapter 13 shows.

EXERCISES

A. COMPUTATION

- 12.1** Find $\lim_{n \rightarrow \infty} \frac{3n^2 - n}{5n^2 + 2}$.
- 12.2** Find $\lim_{n \rightarrow \infty} n \sin \frac{1}{n}$.
- 12.3** Find $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$.
- 12.4** Find $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$.
- 12.5** Determine whether $a_n = (-1)^n \frac{n}{n+1}$ converges.
- 12.6** Find $\lim_{n \rightarrow \infty} \frac{n^2}{e^n}$.
- 12.7** Let $a_1 = 1$ and $a_{n+1} = \frac{a_n + 6}{2}$. Show that (a_n) is increasing and bounded, and find its limit.
- 12.8** Let $a_1 = 2$ and $a_{n+1} = \frac{1}{3 - a_n}$. Show that (a_n) is decreasing and bounded below by 0, and find its limit.
- 12.9** Find $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$.
- 12.10** Find $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$.
- 12.11** Write $0.\overline{123}$ as a fraction in lowest terms.
- 12.12** Find $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.
- 12.13** Show that $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ diverges by finding its partial sums.
- 12.14** Find $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right)$.
- 12.15** Determine whether $\sum_{n=1}^{\infty} \frac{n}{n+1}$ converges.
- 12.16** Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ converges.
- 12.17** Use the integral test to decide whether $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.
- 12.18** Use the integral test to decide whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges, and bound its sum.
- 12.19** Estimate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ with five terms, giving upper and lower bounds from the integral test.
- 12.20** Determine whether $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4 + 3}$ converges.

- 12.21** Determine whether $\sum \frac{\sqrt[n]{n}}{n^2+1}$ converges.
- 12.22** Determine whether $\sum \sin \frac{1}{n}$ converges.
- 12.23** Determine whether $\sum \frac{2^n+n}{3^n-n}$ converges.
- 12.24** Determine whether $\sum \frac{1}{n3^n}$ converges.
- 12.25** Determine whether $\sum \frac{(-1)^n}{\sqrt{n}}$ converges absolutely, conditionally, or not at all.
- 12.26** Determine whether $\sum \frac{(-1)^{nn}}{n^2+1}$ converges absolutely, conditionally, or not at all.
- 12.27** Determine whether $\sum \frac{(-1)^n}{n^2}$ converges absolutely, conditionally, or not at all.
- 12.28** Determine whether $\sum (-1)^n \frac{n}{n+1}$ converges.
- 12.29** Use the ratio test on $\sum \frac{n^3}{3^n}$.
- 12.30** Use the ratio test on $\sum \frac{10^n}{n!}$.
- 12.31** Use the ratio test on $\sum \frac{(2n)!}{(n!)^2 5^n}$.
- 12.32** How many terms of $\sum \frac{(-1)^{n+1}}{n^3}$ guarantee an error below 0.0005?

B. THEORY AND PROOF

- 12.33** Let $r_n = \frac{F_{n+1}}{F_n}$ for the Fibonacci numbers. Show that $r_{n+1} = 1 + \frac{1}{r_n}$, that $|r_{n+1} - \varphi| \leq \frac{|r_n - \varphi|}{\varphi}$ (using $r_n \geq 1$), and deduce that $r_n \rightarrow \varphi$.
- 12.34** Prove that a convergent sequence is bounded, and give a bounded sequence that diverges.
- 12.35** Prove that if $\sum a_n$ converges with $a_n \geq 0$, then $\sum a_n^2$ converges. Show by example that the hypothesis $a_n \geq 0$ is needed.
- 12.36** Prove the *Cauchy condensation test*: if $a_n \geq 0$ is decreasing, then $\sum a_n$ converges if and only if $\sum 2^k a_{2^k}$ does. Use it to give a second proof of Corollary 12.13.
- 12.37** Show that the root test is at least as strong as the ratio test: if $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow L$, then $|a_n|^{1/n} \rightarrow L$.

12.38 Prove that $H_N - \ln N$ is decreasing and bounded below by 0, so Euler's constant $\gamma = \lim(H_N - \ln N)$ exists.

C. CHALLENGE

12.39 Prove that e is irrational. (Suppose $e = \frac{p}{q}$; multiply $e = \sum \frac{1}{n!}$ by $q!$ and show that the remainder is an integer strictly between 0 and 1.)

12.40 A stack of identical blocks of length 2 is built so that each overhangs the one below as far as possible without toppling. Show that the top of a stack of n blocks can overhang the bottom block by $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ in total, so the overhang can be made arbitrarily large.

12.41 Show that $\sum \frac{\sin n}{n}$ converges. (Sum by parts, using the bounded partial sums of $\sin n$.)

12.42 Find all p for which $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converges.

12.43 Rearrange the alternating harmonic series to make its sum 2, describing the pattern of the first few terms.

CHAPTER 13

POWER SERIES AND TAYLOR SERIES

LEARNING OBJECTIVES

- Find the radius and interval of convergence of a power series.
- Represent functions as power series by algebraic manipulation, differentiation, and integration of known series.
- Compute Taylor and Maclaurin series, and know the standard series by heart.
- Prove convergence of a Taylor series to its function with Taylor's theorem, and bound the error of a Taylor polynomial.
- Use series to evaluate limits, integrals, and sums, and to approximate physical formulas; meet Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$.

Polynomials are the easiest functions to compute: only addition and multiplication are needed. This chapter shows that the functions of calculus, e^x , $\sin x$, $\ln(1+x)$, $\arctan x$, are “polynomials of infinite degree” on suitable intervals, and that truncating these series gives polynomial approximations with controlled error. This is how calculators compute, how physicists simplify, and how the linear approximation of Chapter 4 is extended to any order.

13.1 POWER SERIES

Definition 13.1. A power series centered at a is a series $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$, with the convention $(x-a)^0 = 1$ even at $x = a$.

Lemma 13.2 (Abel). If $\sum c_n x^n$ converges at $x = b \neq 0$, it converges absolutely for every $|x| < |b|$.

Proof. Since $\sum c_n b^n$ converges, $c_n b^n \rightarrow 0$, so $|c_n b^n| \leq M$ for all n . Then $|c_n x^n| = |c_n b^n| \left| \frac{x}{b} \right|^n \leq M \left| \frac{x}{b} \right|^n$, a convergent geometric series. ■

Theorem 13.3. For a power series $\sum c_n(x-a)^n$ exactly one of the following holds: it converges only at $x = a$; it converges absolutely for all x ; or there is an $R > 0$, the radius of convergence, such that it converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.

Proof. Take $a = 0$. Let R be the least upper bound of the set of $|x|$ at which the series converges (or ∞ if that set is unbounded). If $|x| < R$, there is a point b of convergence with $|x| < |b|$, and Lemma 13.2 gives absolute convergence at x . If $|x| > R$, the series diverges at x by the definition of R . ■

We set $R = 0$ and $R = \infty$ in the first two cases. At the endpoints $x = a \pm R$ anything can happen, and each must be tested separately. The *interval of convergence* is the set of all x where the series converges. The ratio test usually finds R : if $\left| \frac{c_{n+1}}{c_n} \right| \rightarrow L$, then $R = \frac{1}{L}$.

Example 13.4 (Intervals of convergence). (a) $\sum \frac{x^n}{n!}$: $\frac{|x|}{n+1} \rightarrow 0$, so $R = \infty$. (b) $\sum n! x^n$: the ratio is $(n+1)|x| \rightarrow \infty$ for $x \neq 0$, so $R = 0$. (c) $\sum \frac{(x-2)^n}{n 3^n}$: the ratio tends to $\frac{|x-2|}{3}$, so $R = 3$. At $x = 5$ the series is $\sum \frac{1}{n}$, divergent; at $x = -1$ it is $\sum \frac{(-1)^n}{n}$, convergent. The interval is $[-1, 5)$.

13.2 REPRESENTING FUNCTIONS BY POWER SERIES

The geometric series is the source of many others:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \quad |x| < 1.$$

Substitution and multiplication give more: $\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$ for $|x| < 1$, and $\frac{x}{2-x} = \frac{x}{2} \cdot \frac{1}{1-x/2} = \sum \frac{x^{n+1}}{2^{n+1}}$ for $|x| < 2$.

Theorem 13.5 (Termwise differentiation and integration). If $f(x) = \sum c_n(x-a)^n$ has radius of convergence $R > 0$, then f is differentiable on

$(a - R, a + R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}, \quad \int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x - a)^{n+1},$$

both series having radius of convergence R .

The proof, which needs uniform convergence, is in Appendix A. The radius is preserved because $n^{1/n} \rightarrow 1$, but endpoint behavior can change.

Example 13.6 (Logarithm and arctangent). Integrate $\frac{1}{1+t} = \sum (-1)^n t^n$ and $\frac{1}{1+t^2} = \sum (-1)^n t^{2n}$ from 0 to x :

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \end{aligned}$$

for $|x| < 1$. Both series also converge at $x = 1$, by the alternating series test, and *Abel's theorem* (a power series that converges at an endpoint is continuous there from inside) gives

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots, \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

The second is the *Leibniz series* (1674), beautiful but useless for computation: by Theorem 12.18, five decimal places of $\frac{\pi}{4}$ need 100 000 terms.

Example 13.7 (Summing a series by differentiating). Differentiate $\sum_{n \geq 0} x^n = \frac{1}{1-x}$ and multiply by x : $\sum n x^n = \frac{x}{(1-x)^2}$. Repeating, $\sum n^2 x^n = x \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}$. At $x = \frac{1}{2}$, $\sum \frac{n^2}{2^n} = \frac{(1/2)(3/2)}{1/8} = 6$, the value promised in Chapter 12.

13.3 TAYLOR AND MACLAURIN SERIES

If $f(x) = \sum c_n (x - a)^n$ on an interval about a , differentiating k times and setting $x = a$ kills every term except the k th: $f^{(k)}(a) = k! c_k$. So the coefficients are determined by f :

Theorem 13.8. If f has a power series representation at a with $R > 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

This is the *Taylor series* of f at a ; when $a = 0$ it is the *Maclaurin series*. Its partial sum $T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x - a)^n$ is the *Taylor polynomial* of degree N . T_1 is the linear approximation; T_N is the unique polynomial of degree at most N that agrees with f and its first N derivatives at a .

The theorem says that if a series exists, it must be the Taylor series. It does not say that the Taylor series converges to f .

Example 13.9 (A function not equal to its Taylor series). Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Every derivative of f at 0 is 0 (each is a limit of the form $\lim_{x \rightarrow 0} \frac{p(1/x)}{e^{1/x^2}} = 0$ for a polynomial p), so its Maclaurin series is identically zero, and converges everywhere, but equals $f(x)$ only at $x = 0$. The function is so flat at 0 that no polynomial detects it.

13.4 TAYLOR'S THEOREM

To show a Taylor series converges to its function, we must show the *remainder* $R_N(x) = f(x) - T_N(x)$ tends to 0.

Theorem 13.10 (Taylor's theorem). If $f^{(N+1)}$ is continuous on an interval containing a and x , then

$$R_N(x) = \frac{1}{N!} \int_a^x (x - t)^N f^{(N+1)}(t) dt = \frac{f^{(N+1)}(c)}{(N+1)!} (x - a)^{N+1}$$

for some c between a and x . In particular, if $|f^{(N+1)}| \leq M$ between a and x , then $|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!}$.

Proof. For $N = 0$, the integral formula is the Fundamental Theorem: $f(x) - f(a) = \int_a^x f'(t) dt$. Assume it for $N - 1$ and integrate by parts with $u = f^{(N)}(t)$,

$$dv = \frac{(x-t)^{N-1}}{(N-1)!} dt, v = -\frac{(x-t)^N}{N!}:$$

$$\begin{aligned} R_{N-1}(x) &= \int_a^x \frac{(x-t)^{N-1}}{(N-1)!} f^{(N)}(t) dt \\ &= \frac{f^{(N)}(a)}{N!} (x-a)^N + \int_a^x \frac{(x-t)^N}{N!} f^{(N+1)}(t) dt. \end{aligned}$$

Since $R_N = R_{N-1} - \frac{f^{(N)}(a)}{N!} (x-a)^N$, this is the integral formula for R_N . For the second form, take $a < x$ (the other case is similar). The weight $w(t) = \frac{(x-t)^N}{N!}$ is nonnegative on $[a, x]$, with $\int_a^x w = \frac{(x-a)^{N+1}}{(N+1)!}$. If m and M are the minimum and maximum of $f^{(N+1)}$ on $[a, x]$, then $m \int w \leq \int w f^{(N+1)} \leq M \int w$, so $\int w f^{(N+1)} / \int w$ lies between m and M , and by the Intermediate Value Theorem it equals $f^{(N+1)}(c)$ for some c in $[a, x]$. ■

The second form, due to Lagrange, says that the error looks like the next term of the series, with the derivative evaluated at an unknown point.

Theorem 13.11 (Standard series). For all real x ,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Proof. Every derivative of e^x is e^x , bounded by $e^{|x|}$ between 0 and x , so $|R_N(x)| \leq e^{|x|} \frac{|x|^{N+1}}{(N+1)!} \rightarrow 0$ by Example 12.2(e). Every derivative of $\sin$ and $\cos$ is $\pm \sin$ or $\pm \cos$, bounded by 1, so $|R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!} \rightarrow 0$. The coefficients follow from $f^{(n)}(0)$: all 1 for e^x , and the cycle 0, 1, 0, -1 for $\sin$. ■

Function	Maclaurin series	Valid for
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \dots$	$ x < 1$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	all x
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	all x
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$	$-1 < x \leq 1$
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	$-1 \leq x \leq 1$
$(1+x)^k$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$	$ x < 1$

Example 13.12 (Computing e and $\sin 0.1$). With $N = 10$, $e \approx \sum_{n=0}^{10} \frac{1}{n!} = 2.718281801$, with error at most $\frac{e}{11!} < \frac{3}{11!} \approx 7.5 \times 10^{-8}$; the true value is 2.718281828. For $\sin 0.1$, the series alternates, so $0.1 - \frac{0.001}{6} = 0.0998333$ has error at most $\frac{0.1^5}{120} \approx 8.3 \times 10^{-8}$; indeed $\sin 0.1 = 0.09983342$.

COMMON ERROR. A Taylor polynomial is accurate near its center and can be useless far from it. T_{10} for e^x at 0 gives e^{10} with an error of more than 40%, and T_N for $\ln(1+x)$ diverges for every $x > 1$ no matter how large N is. Choose the center near the point of interest, or transform the problem first.

13.4.1 THE BINOMIAL SERIES

Theorem 13.13 (Binomial series). For every real k and $|x| < 1$,

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad \binom{k}{n} = \frac{k(k-1) \cdots (k-n+1)}{n!}.$$

Proof. The ratio test gives $R = 1$ (the ratio of coefficients is $\frac{k-n}{n+1} \rightarrow -1$). Let $g(x)$ be the sum. Differentiating termwise and using $(n+1)\binom{k}{n+1} + n\binom{k}{n} = k\binom{k}{n}$, one checks $(1+x)g'(x) = kg(x)$. Then $\frac{d}{dx}[(1+x)^{-k}g(x)] = (1+x)^{-k-1}((1+x)g' - kg) = 0$, so $(1+x)^{-k}g(x) = g(0) = 1$. ■

When k is a nonnegative integer, the coefficients vanish for $n > k$ and this is the binomial theorem. Newton discovered the general case in 1665.

Example 13.14 (Square roots). $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$ and $\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots$. For example, $\sqrt{1.1} \approx 1 + 0.05 - 0.00125 = 1.04875$, compared with 1.048809.

13.5 APPLICATIONS OF TAYLOR SERIES

13.5.1 LIMITS

Series show the dominant behavior of a function near a point at once. Since $\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \cdots$,

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \left(-\frac{1}{6} + \frac{x^2}{120} - \cdots \right) = -\frac{1}{6},$$

which L'Hôpital's rule reaches only after three differentiations. Similarly $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$ and $\lim_{x \rightarrow 0} \frac{\arctan x - x}{x^3} = -\frac{1}{3}$ can be read off the series.

13.5.2 INTEGRALS

Example 13.15 (The error function). Evaluate $\int_0^1 e^{-x^2} dx$ to four decimal places.

Solution. e^{-x^2} has no elementary antiderivative, but $e^{-x^2} = \sum \frac{(-1)^n x^{2n}}{n!}$, and integrating termwise,

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (2n+1)} \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360} - \cdots \end{aligned}$$

This is alternating with decreasing terms. The eighth term is $\frac{1}{5040 \cdot 15} \approx 1.3 \times 10^{-5}$, so the sum of the first seven terms, 0.746836, has error less than 1.3×10^{-5} , and the integral is 0.7468 to four places. This integral is the basis of the normal distribution in statistics.

13.5.3 PRODUCTS AND QUOTIENTS

Series can be multiplied like polynomials, collecting powers: $e^x \sin x = (1 + x + \frac{x^2}{2} + \cdots)(x - \frac{x^3}{6} + \cdots) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$. Division works too: from $\tan x \cos x = \sin x$, matching coefficients gives $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$.

13.5.4 EULER'S FORMULA

Substituting $x = i\theta$, with $i^2 = -1$, into the series for e^x and separating real and imaginary parts (legitimate for absolutely convergent series):

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \right) + i \left(\theta - \frac{\theta^3}{3!} + \cdots \right) = \cos \theta + i \sin \theta.$$

We take this as the definition of the complex exponential. At $\theta = \pi$ it gives $e^{i\pi} + 1 = 0$, linking the five most important constants of mathematics.

LOOKING AHEAD. Euler's formula is the language of quantum mechanics. In Volume IV, a free particle has wave function $e^{i(kx - \omega t)}$, and the time evolution of every quantum state is multiplication by $e^{-iEt/\hbar}$. Taylor series in several variables, and the expansion of a potential about its minimum, $V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2$, explain why the harmonic oscillator appears everywhere in physics.

13.6 WORKED PROBLEMS

Each problem below is solved in full and combines several ideas from the chapter. Attempt each one before reading its solution.

— Problem 13.1 Euler and the Basel problem

Reproduce Euler's 1735 argument that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, and say what it lacks as a proof.

Solution. A polynomial with roots $r_1, \dots, r_k$ and value 1 at 0 factors as $\prod (1 - \frac{x}{r_i})$. The function $\frac{\sin x}{x}$ has value 1 at 0 and roots $\pm\pi, \pm 2\pi, \dots$; treating it as a polynomial of infinite degree,

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \cdots$$

The coefficient of x^2 on the right is $-\frac{1}{\pi^2} \sum \frac{1}{n^2}$; on the left, from $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \dots$, it is $-\frac{1}{6}$. Equating, $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. The numerics agree: $\sum_{n \leq 100} \frac{1}{n^2} \approx 1.63498$, and the integral test bounds the remainder between $\frac{1}{101}$ and $\frac{1}{100}$, consistent with $\frac{\pi^2}{6} \approx 1.64493$.

The gap is the infinite product: a function is not determined by its zeros ($e^{x \frac{\sin x}{x}}$ has the same zeros). Weierstrass justified this product a century later. In Volume IV, Fourier series give a two-line proof.

— Problem 13.2 Relativistic kinetic energy

In special relativity a particle of mass m and speed v has kinetic energy $K = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right)$. Show that $K \approx \frac{1}{2}mv^2$ for $v \ll c$, and estimate the relative error of the classical formula at $v = 0.1c$.

Solution. With $\beta = \frac{v}{c}$ and the binomial series $(1 - u)^{-1/2} = 1 + \frac{u}{2} + \frac{3u^2}{8} + \cdots$ at $u = \beta^2$,

$$K = mc^2 \left(\frac{\beta^2}{2} + \frac{3\beta^4}{8} + \frac{5\beta^6}{16} + \cdots \right) = \frac{1}{2}mv^2 \left(1 + \frac{3}{4}\beta^2 + \cdots \right).$$

The leading term is Newton's kinetic energy. At $\beta = 0.1$ the correction is about $\frac{3}{4}(0.01) = 0.75\%$. For a car at 30 m/s, $\beta^2 = 10^{-14}$: Newtonian mechanics is the first term of a series whose second term is invisible in daily life.

— Problem 13.3 Computing π

Show that $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$ (Machin, 1706), and estimate how many terms of each arctangent series give π to 7 decimal places.

Solution. Let $\alpha = \arctan \frac{1}{5}$. By the double angle formula, $\tan 2\alpha = \frac{2/5}{1-1/25} = \frac{5}{12}$ and $\tan 4\alpha = \frac{10/12}{1-25/144} = \frac{120}{119}$. Then

$$\tan \left(4\alpha - \frac{\pi}{4} \right) = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{1}{239},$$

and since $4\alpha - \frac{\pi}{4}$ is between 0 and $\frac{\pi}{2}$, it equals $\arctan \frac{1}{239}$. Both series alternate. Five terms of $\arctan \frac{1}{5}$ have error at most $\frac{(1/5)^{11}}{11} \approx 1.9 \times 10^{-9}$, multiplied by 16 in π ; two terms of $\arctan \frac{1}{239}$ have error at most $\frac{(1/239)^5}{5}$, negligible. The total error is under 4×10^{-8} , and the computation gives 3.14159268, against $\pi = 3.14159265$. Machin computed 100 digits by hand this way; formulas of this type held every record until the 1970s.

— Problem 13.4 Choosing a center

Approximate $\sqrt[3]{8.1}$ with a second-degree Taylor polynomial, and bound the error.

Solution. Expand $f(x) = x^{1/3}$ at $a = 8$, where everything is exact: $f(8) = 2$, $f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{12}$, $f''(8) = -\frac{2}{9} \cdot 8^{-5/3} = -\frac{1}{144}$. So

$$T_2(8.1) = 2 + \frac{0.1}{12} - \frac{(0.1)^2}{288} \approx 2.0082986.$$

By Theorem 13.10, $|R_2| \leq \frac{M(0.1)^3}{6}$, where M bounds $f'''(x) = \frac{10}{27}x^{-8/3}$ on $[8, 8.1]$; its maximum is at $x = 8$, $\frac{10}{27 \cdot 256} \approx 0.00145$. So the error is at most 2.4×10^{-7} , and in fact it is 2.39×10^{-7} : the Lagrange bound is nearly sharp here because f''' changes little over the interval. Expanding at 0 would be hopeless, since $x^{1/3}$ is not differentiable there.

— **Problem 13.5** Series solution of a differential equation

Find the power series solution of $y' = x + y$, $y(0) = 1$, and identify it.

Solution. Put $y = \sum c_n x^n$ with $c_0 = 1$. Then $y' = \sum (n+1)c_{n+1}x^n$, and matching coefficients of x^n in $y' = x + y$:

$$(n+1)c_{n+1} = c_n + [n = 1],$$

where $[n = 1]$ is 1 if $n = 1$ and 0 otherwise. So $c_1 = 1$, $c_2 = \frac{1+1}{2} = 1$, $c_3 = \frac{1}{3}$, $c_4 = \frac{1}{12}$, and for $n \geq 2$, $c_n = \frac{2}{n!}$. Hence $y = 1 + x + 2 \sum_{n \geq 2} \frac{x^n}{n!} = 1 + x + 2(e^x - 1 - x) = 2e^x - x - 1$, the solution found by Euler's method in Example 10.2. Many equations of physics, such as Bessel's and Hermite's equations in Volume IV, have no elementary solutions, and their power series solutions define new functions.

— **Problem 13.6** Taylor series of the logarithm at 2

Find the Taylor series of $\ln x$ at $a = 2$ and its interval of convergence.

Solution. Write $\ln x = \ln 2 + \ln(1 + \frac{x-2}{2})$ and substitute $u = \frac{x-2}{2}$ in the series for $\ln(1+u)$:

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n 2^n}, \quad -1 < \frac{x-2}{2} \leq 1 \iff 0 < x \leq 4.$$

The interval is centered at 2 and reaches exactly to 0, where $\ln$ fails to exist. This is no accident: a power series converges in the largest interval (in fact disc, in the complex plane) that avoids the singularities of its function, which is why $\frac{1}{1+x^2}$, with no real singularity, still has $R = 1$ at 0: it is singular at $x = \pm i$.

EXERCISES

A. COMPUTATION

13.1 Find the interval of convergence of $\sum \frac{x^n}{n^2}$.

13.2 Find the interval of convergence of $\sum nx^n$.

13.3 Find the interval of convergence of $\sum \frac{(x+1)^n}{2^n \sqrt{n}}$.

13.4 Find the interval of convergence of $\sum n!(x-1)^n$.

13.5 Find the interval of convergence of $\sum \frac{(-1)^n x^{2n}}{(2n)!}$.

- 13.6** Find the interval of convergence of $\sum \frac{(3x-2)^n}{n}$.
- 13.7** Find the radius of convergence of $\sum n^n x^n$.
- 13.8** Find the interval of convergence of $\sum_{n \geq 2} \frac{x^n}{\ln n}$.
- 13.9** Find a power series for $\frac{1}{1+3x}$ and its radius of convergence.
- 13.10** Find a power series for $\frac{x^2}{1-x^4}$.
- 13.11** Find a power series for $\ln(1-x^2)$ and its radius of convergence.
- 13.12** Differentiate a geometric series to find a power series for $\frac{1}{(1-x)^2}$.
- 13.13** Find a power series for $x \arctan(x^2)$ and its radius of convergence.
- 13.14** Find $\sum_{n=1}^{\infty} \frac{n}{3^n}$.
- 13.15** Find $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n}$.
- 13.16** Find the Taylor series of e^x at $a = 3$.
- 13.17** Find the Taylor series of $\cos x$ at $a = \frac{\pi}{2}$.
- 13.18** Find the Maclaurin series of $\sinh x$ and $\cosh x$.
- 13.19** Find the Maclaurin series of $x^2 e^{-x}$.
- 13.20** Find the Maclaurin series of $\sin^2 x$.
- 13.21** Find the first three nonzero terms of the Maclaurin series of $\sec x$.
- 13.22** Find T_3 for $\sqrt{x}$ at $a = 4$.
- 13.23** Find T_3 for $\ln x$ at $a = 1$, evaluate it at 1.2, and bound the error with Theorem 13.10.
- 13.24** What degree of Taylor polynomial at 0 approximates $e^{0.5}$ within 10^{-6} ?
- 13.25** Use the binomial series to approximate $\sqrt[3]{1.06}$ to four decimal places.
- 13.26** Find the binomial series for $\frac{1}{(1+x)^3}$.
- 13.27** Use series to find $\lim_{x \rightarrow 0} \frac{1 - \cos x - x^2/2}{x^4}$.
- 13.28** Use series to find $\lim_{x \rightarrow 0} \frac{e^x - 1 - x - x^2/2}{x^3}$.
- 13.29** Evaluate $\int_0^1 \sin(x^2) dx$ to three decimal places.

13.30 Evaluate $\int_0^{1/2} \frac{\ln(1+x)}{x} dx$ to three decimal places.

13.31 Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^n (2n)!}$.

13.32 Write $e^{i\pi/3}$ and $e^{i\pi/2}$ in the form $a + bi$.

B. THEORY AND PROOF

13.33 Prove that if $\sum c_n x^n$ has radius R , then $\sum c_n x^{2n}$ has radius $\sqrt{R}$.

13.34 Prove that the Maclaurin series of f is even (only even powers) when f is even.

13.35 Prove that if $|c_n|^{1/n} \rightarrow L$ with $0 < L < \infty$, the radius of convergence is $\frac{1}{L}$.

13.36 Use Taylor's theorem to prove the second derivative test: if $f'(a) = 0$ and $f''(a) > 0$, with f'' continuous, then f has a local minimum at a .

13.37 Prove that $|\sin x - x| \leq \frac{|x|^3}{6}$ for all x .

13.38 Use Euler's formula to derive the addition formulas for $\cos(a+b)$ and $\sin(a+b)$ from $e^{i(a+b)} = e^{ia}e^{ib}$.

C. CHALLENGE

13.39 Prove that every derivative of e^{-1/x^2} at 0 is 0, completing Example 13.9.

13.40 The period of a pendulum of length L released from angle θ_0 is $T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$, $k = \sin \frac{\theta_0}{2}$. Expand by the binomial series to show $T \approx 2\pi\sqrt{\frac{L}{g}} \left(1 + \frac{\theta_0^2}{16}\right)$.

13.41 Show that $\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} = 5e$.

13.42 Find the Maclaurin series of $\frac{x}{e^x - 1}$ up to x^4 . Its coefficients define the Bernoulli numbers, which give the sums $\sum \frac{1}{n^{2k}}$.

13.43 Prove Abel's theorem for the series of $\ln(1+x)$ directly: show that $\left| \ln 2 - \sum_{n=1}^N \frac{(-1)^{n-1}}{n} \right| \leq \int_0^1 t^N dt$, using $\frac{1}{1+t} = \sum_{n < N} (-t)^n + \frac{(-t)^N}{1+t}$.

APPENDIX A

FOUNDATIONS: PROOFS OF THE THEOREMS ON CONTINUITY, INTEGRABILITY, AND SERIES

This appendix proves the theorems whose proofs were postponed. Everything rests on the completeness axiom of Chapter 1: *every nonempty set of real numbers that is bounded above has a least upper bound*. The least upper bound of S is written $\sup S$; the greatest lower bound, which exists for nonempty sets bounded below (apply the axiom to $-S$), is written $\inf S$. The defining property used again and again: if $s = \sup S$ and $\epsilon > 0$, then $s - \epsilon$ is not an upper bound, so some element of S exceeds $s - \epsilon$.

A.1 CONSEQUENCES OF COMPLETENESS

Theorem A.1 (Archimedean property). *For every real x there is an integer $n > x$. Consequently $\frac{1}{n} \rightarrow 0$.*

Proof. If not, x is an upper bound of $\mathbb{N}$, so $s = \sup \mathbb{N}$ exists. Then $s - 1$ is not an upper bound: some $n > s - 1$. But then $n + 1 > s$ and $n + 1 \in \mathbb{N}$, a contradiction. ■

Theorem A.2 (Bolzano–Weierstrass). *Every bounded sequence has a convergent subsequence.*

Proof. Let $a \leq x_n \leq b$ for all n . Bisect $[a, b]$; at least one half contains x_n for infinitely many n . Call it $[a_1, b_1]$, and choose n_1 with $x_{n_1} \in [a_1, b_1]$. Bisect again, choose a half $[a_2, b_2]$ containing infinitely many terms, and choose $n_2 > n_1$ with $x_{n_2} \in [a_2, b_2]$. Continuing, we obtain nested intervals with $b_k - a_k = \frac{b-a}{2^k}$ and indices $n_1 < n_2 < \dots$ with $x_{n_k} \in [a_k, b_k]$. The sequence (a_k) is increasing and

bounded by b , so it converges to some c by Theorem 12.4 (whose proof uses only the completeness axiom). Since $a_k \leq c \leq b_k$, $|x_{n_k} - c| \leq b_k - a_k \rightarrow 0$. ■

We also use the *sequential criterion*: f is continuous at c if and only if $f(x_n) \rightarrow f(c)$ for every sequence $x_n \rightarrow c$ in the domain. The forward direction is immediate from the definitions. Conversely, if f is not continuous at c , there is an $\epsilon > 0$ such that for every n some x_n with $|x_n - c| < \frac{1}{n}$ has $|f(x_n) - f(c)| \geq \epsilon$; then $x_n \rightarrow c$ but $f(x_n) \not\rightarrow f(c)$.

A.2 THE INTERMEDIATE AND EXTREME VALUE THEOREMS

Proof of the Intermediate Value Theorem (Theorem 2.28). Let f be continuous on $[a, b]$ and $f(a) < N < f(b)$ (the case $f(a) > N > f(b)$ follows by applying this to $-f$). Let $S = \{x \in [a, b] : f(x) < N\}$, nonempty (it contains a) and bounded by b ; let $c = \sup S$. We show $f(c) = N$.

If $f(c) < N$, then $c < b$, and by continuity $f(x) < N$ on some interval $[c, c + \delta)$, which contains points of S larger than c : impossible. If $f(c) > N$, then $c > a$, and by continuity $f(x) > N$ on some interval $(c - \delta, c]$. No point of this interval is in S , and no point to the right of c is in S , so $c - \delta$ is an upper bound of S smaller than c : impossible. Hence $f(c) = N$. ■

Proof of the Extreme Value Theorem (Theorem 2.30). Let f be continuous on $[a, b]$. First, f is bounded: otherwise for each n there is x_n with $|f(x_n)| > n$. By Theorem A.2, a subsequence $x_{n_k} \rightarrow c \in [a, b]$, and by continuity $f(x_{n_k}) \rightarrow f(c)$, a finite number, contradicting $|f(x_{n_k})| > n_k \rightarrow \infty$.

Let $M = \sup\{f(x) : a \leq x \leq b\}$. For each n , $M - \frac{1}{n}$ is not an upper bound, so there is x_n with $f(x_n) > M - \frac{1}{n}$. A subsequence converges to some $c \in [a, b]$, and $f(c) = \lim f(x_{n_k}) = M$. So the maximum is attained; apply this to $-f$ for the minimum. ■

A.3 INVERSE FUNCTIONS AND THE ROOT LAW

Theorem A.3. Let f be continuous and one-to-one on an interval I . Then f is strictly monotone, $f(I)$ is an interval, and f^{-1} is continuous on $f(I)$.

Proof. Monotone. If not, there are $x < y < z$ in I with $f(y)$ not between $f(x)$ and

$f(z)$; say $f(y) > \max(f(x), f(z))$ (the other case is symmetric). Any value N strictly between $\max(f(x), f(z))$ and $f(y)$ is taken on (x, y) and on (y, z) , by the Intermediate Value Theorem, contradicting one-to-one.

Interval. By the Intermediate Value Theorem, $f(I)$ contains every value between any two of its values.

Continuity of the inverse. Let f be increasing (so $g = f^{-1}$ is increasing too), $d = f(c)$ an interior point of $f(I)$, and $\epsilon > 0$ small enough that $c \pm \epsilon \in I$. Put $\delta = \min(d - f(c - \epsilon), f(c + \epsilon) - d) > 0$. If $|y - d| < \delta$, then $f(c - \epsilon) < y < f(c + \epsilon)$, and since g is increasing, $c - \epsilon < g(y) < c + \epsilon$. Endpoints are handled with one-sided versions of the same argument. ■

Corollary A.4 (Root law). If $\lim_{x \rightarrow a} f(x) = L$ and n is a positive integer, then $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, provided $L > 0$ when n is even.

Proof. The function $x \mapsto x^n$ is continuous and one-to-one on $[0, \infty)$ (and on $\mathbb{R}$ if n is odd), so by Theorem A.3 its inverse $x \mapsto \sqrt[n]{x}$ is continuous. A continuous function of a function with a limit has the corresponding limit (Chapter 2). If n is even and $L > 0$, then $f(x) > 0$ near a , so $\sqrt[n]{f(x)}$ is defined there. ■

By the same theorem, $\ln$ (the inverse of $\exp$), $\arcsin$, $\arctan$, and the other inverse trigonometric functions are continuous, which completes the proof of the continuity theorem of Chapter 2.

A.4 L'HÔPITAL'S RULE: THE REMAINING CASES

Proof of the $\frac{\infty}{\infty}$ case of Theorem 6.1. Consider $x \rightarrow a^+$ with $g(x) \rightarrow \pm\infty$ and $\frac{f'}{g'} \rightarrow L$, L finite. Let $\epsilon > 0$, and choose $\delta > 0$ with $|\frac{f'(t)}{g'(t)} - L| < \epsilon$ for $a < t < a + \delta$. Fix $y = a + \delta$ (shrinking δ if needed so that f and g are differentiable on $(a, y]$). For $a < x < y$, Cauchy's Mean Value Theorem gives $c \in (x, y)$ with

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}, \quad \text{so} \quad \left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| < \epsilon$$

($g(x) \neq g(y)$) by Rolle's theorem, since $g' \neq 0$). Now write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(y)}{g(x) - g(y)} \cdot \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}.$$

As $x \rightarrow a^+$, $g(x) \rightarrow \pm\infty$ while y is fixed, so the second factor tends to 1 and the last term to 0. Hence $\frac{f(x)}{g(x)}$ lies within 2ϵ of L for x close enough to a . As ϵ

was arbitrary, $\frac{f(x)}{g(x)} \rightarrow L$. (The hypothesis $f \rightarrow \pm\infty$ was not even used.) For $L = +\infty$, the same argument with $\frac{f'}{g} > M$ in place of $|\frac{f'}{g} - L| < \epsilon$ shows $\frac{f}{g} > \frac{M}{2}$ eventually. ■

Proof of the case $a = \infty$. Put $F(t) = f(\frac{1}{t})$ and $G(t) = g(\frac{1}{t})$ for $t > 0$. Then $F'(t) = -\frac{1}{t^2}f'(\frac{1}{t})$ and $G'(t) = -\frac{1}{t^2}g'(\frac{1}{t})$, so $\frac{F'(t)}{G'(t)} = \frac{f'(1/t)}{g'(1/t)} \rightarrow L$ as $t \rightarrow 0^+$. By the cases already proved, $\frac{F(t)}{G(t)} \rightarrow L$ as $t \rightarrow 0^+$, that is, $\frac{f(x)}{g(x)} \rightarrow L$ as $x \rightarrow \infty$. ■

A.5 INTEGRABILITY OF CONTINUOUS FUNCTIONS

Definition A.5. f is *uniformly continuous* on a set S if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S$ with $|x - y| < \delta$.

Theorem A.6 (Heine). *A function continuous on $[a, b]$ is uniformly continuous there.*

Proof. If not, there is $\epsilon > 0$ such that for each n there are $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$. By Theorem A.2, a subsequence $x_{n_k} \rightarrow c \in [a, b]$; then $y_{n_k} \rightarrow c$ too. By continuity, $f(x_{n_k}) - f(y_{n_k}) \rightarrow f(c) - f(c) = 0$, contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$. ■

For a bounded f and a partition P , let M_i and m_i be the supremum and infimum of f on $[x_{i-1}, x_i]$, and define the *upper* and *lower sums* $U(P) = \sum M_i \Delta x_i$ and $L(P) = \sum m_i \Delta x_i$. Every Riemann sum for P lies between $L(P)$ and $U(P)$.

Lemma A.7. *For any two partitions P and Q , $L(P) \leq U(Q)$.*

Proof. Adding a point to a partition splits one subinterval in two; the infimum on each piece is at least the infimum on the whole, so lower sums do not decrease, and similarly upper sums do not increase. Let $R = P \cup Q$, a refinement of both. Then $L(P) \leq L(R) \leq U(R) \leq U(Q)$. ■

Proof of Theorem 7.3 for continuous f . By the Extreme Value Theorem f is bounded, and by Lemma A.7 the set of all lower sums is bounded above by any upper sum. Let $I = \sup_P L(P)$. Then $L(P) \leq I \leq U(P)$ for every partition P .

Given $\epsilon > 0$, by Theorem A.6 choose δ with $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ when $|x - y| < \delta$. If $\|P\| < \delta$, then on each subinterval $M_i - m_i \leq \frac{\epsilon}{b-a}$ (the extremes are attained, by the Extreme Value Theorem, at points less than δ apart), so $U(P) - L(P) \leq \frac{\epsilon}{b-a} \sum \Delta x_i = \epsilon$. Both I and every Riemann sum for P lie in the interval $[L(P), U(P)]$, of length at most ϵ , so they differ by at most ϵ . This is the definition of $\int_a^b f = I$. ■

Sketch for bounded f with finitely many discontinuities. Let $|f| \leq M$ and let the discontinuities be $d_1, \dots, d_k$. Given ϵ , enclose each d_j in an open interval of length $\eta = \frac{\epsilon}{8kM}$. On the complement, a finite union of closed intervals, f is continuous and so uniformly continuous; the argument above makes the contribution of those subintervals to $U(P) - L(P)$ at most $\frac{\epsilon}{2}$ for fine partitions. Subintervals of a fine partition that meet the small intervals have total length at most $2k\eta$, and contribute at most $2M \cdot 2k\eta = \frac{\epsilon}{2}$. So $U(P) - L(P) \leq \epsilon$ when $\|P\|$ is small, and the conclusion follows as before. ■

A.6 UNIFORM CONVERGENCE AND POWER SERIES

Definition A.8. Functions f_n converge uniformly to f on a set S if for every $\epsilon > 0$ there is an N with $|f_n(x) - f(x)| < \epsilon$ for all $n > N$ and all $x \in S$.

Pointwise convergence lets N depend on x ; uniform convergence does not. The functions x^n on $[0, 1]$ converge pointwise to a function that is 0 for $x < 1$ and 1 at $x = 1$: continuous functions with a discontinuous limit, which uniform convergence forbids.

Theorem A.9. Let $f_n \rightarrow f$ uniformly on $[a, b]$, with each f_n continuous. Then f is continuous, and $\int_a^b f_n \rightarrow \int_a^b f$.

Proof. Given ϵ , choose n with $|f_n - f| < \frac{\epsilon}{3}$ on $[a, b]$, and δ with $|f_n(x) - f_n(c)| < \frac{\epsilon}{3}$ for $|x - c| < \delta$. Then $|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| < \epsilon$. For the integrals, f is continuous and hence integrable, and $|\int_a^b (f_n - f)| \leq \int_a^b |f_n - f| \leq (b - a) \sup |f_n - f| \rightarrow 0$. ■

Theorem A.10 (Weierstrass M -test). If $|u_n(x)| \leq M_n$ on S and $\sum M_n$ converges, then $\sum u_n(x)$ converges absolutely and uniformly on S .

Proof. Absolute convergence at each x is the comparison test. The remainder after N terms satisfies $|\sum_{n>N} u_n(x)| \leq \sum_{n>N} M_n$, which tends to 0 independently of x . ■

Proof of Theorem 13.5. Take $a = 0$ and $0 < r < R$. Choose s with $r < s < R$; the series converges at s , so $|c_n|s^n \leq K$ for all n . On $[-r, r]$,

$$|c_n x^n| \leq K \left(\frac{r}{s}\right)^n, \quad |nc_n x^{n-1}| \leq \frac{K}{s} n \left(\frac{r}{s}\right)^{n-1},$$

and both bounds form convergent series (the second by the ratio test). By the M -test, both $\sum c_n x^n$ and $g(x) = \sum nc_n x^{n-1}$ converge uniformly on $[-r, r]$, and their sums are continuous by Theorem A.9.

Integration. Applying Theorem A.9 to the partial sums on the interval between 0 and x gives $\int_0^x f = \sum c_n \frac{x^{n+1}}{n+1}$.

Differentiation. Integrating g termwise in the same way, $\int_0^x g(t) dt = \sum_{n \geq 1} c_n x^n = f(x) - c_0$. Since g is continuous, the Fundamental Theorem gives $f'(x) = g(x)$ for $|x| < r$, and $r < R$ was arbitrary.

Radii. The argument shows the differentiated and integrated series converge for $|x| < R$. If either converged at some $|x| > R$, then differentiating or integrating it back would give convergence of the original series beyond R , by what was just proved. ■

Theorem A.11 (Abel). *If $\sum c_n$ converges to s , then $\sum c_n x^n$ converges for $|x| < 1$ and $\lim_{x \rightarrow 1^-} \sum c_n x^n = s$.*

Proof. Convergence for $|x| < 1$ is Lemma 13.2. Let $s_n = c_0 + \cdots + c_n$. For $|x| < 1$, the identity $\sum_{n \geq 0} c_n x^n = (1-x) \sum_{n \geq 0} s_n x^n$ follows by multiplying out (both series converge absolutely, since (s_n) is bounded). Since $(1-x) \sum x^n = 1$,

$$\sum c_n x^n - s = (1-x) \sum_{n=0}^{\infty} (s_n - s) x^n.$$

Given $\epsilon > 0$, choose N with $|s_n - s| < \epsilon$ for $n > N$. For $0 < x < 1$, the right side is at most $(1-x) \sum_{n \leq N} |s_n - s| + \epsilon(1-x) \sum_{n > N} x^n \leq (1-x)C + \epsilon$, where C does not depend on x . This is less than 2ϵ for x close to 1. ■

APPENDIX B

FORMULAS AND TABLES

Every antiderivative below has been checked by differentiation. Constants of integration are omitted, and $a > 0$ throughout.

ALGEBRA AND GEOMETRY

Quadratic formula	$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Factoring	$a^2 - b^2 = (a - b)(a + b), \quad a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Binomial theorem	$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$
Exponents	$a^m a^n = a^{m+n}, \quad (a^m)^n = a^{mn}$ $a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = \sqrt[n]{a^m}$
Logarithms	$\ln(xy) = \ln x + \ln y, \quad \ln \frac{x}{y} = \ln x - \ln y$ $\ln x^r = r \ln x, \quad \log_b x = \frac{\ln x}{\ln b}$
Distance, midpoint	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \quad \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$
Circle	area πr^2 , circumference $2\pi r$; sector $\frac{1}{2} r^2 \theta$, arc $r\theta$
Sphere	volume $\frac{4}{3} \pi r^3$, area $4\pi r^2$
Cylinder	volume $\pi r^2 h$
Cone	volume $\frac{1}{3} \pi r^2 h$, lateral area $\pi r \sqrt{r^2 + h^2}$

TRIGONOMETRY

Pythagorean	$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$
Addition	$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$ $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ $\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$
Double angle	$\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$
Half angle	$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$
Product to sum	$\sin a \cos b = \frac{1}{2} [\sin(a - b) + \sin(a + b)]$ $\sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$ $\cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$
Law of sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
Law of cosines	$c^2 = a^2 + b^2 - 2ab \cos C$
Hyperbolic	$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \cosh^2 x - \sinh^2 x = 1$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	—	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

DERIVATIVES

f	f'	f	f'
x^r	rx^{r-1}	e^x, a^x	$e^x, a^x \ln a$
$\ln x $	$\frac{1}{x}$	$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{arcsec} x$	$\frac{1}{ x \sqrt{x^2-1}}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Rules: $(cf)' = cf'$, $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, $(f \circ g)' = (f' \circ g)g'$, $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.

INTEGRALS

$\int x^r dx = \frac{x^{r+1}}{r+1} \quad (r \neq -1)$	$\int \frac{dx}{x} = \ln x $
$\int e^x dx = e^x$	$\int a^x dx = \frac{a^x}{\ln a} \quad (a \neq 1)$
$\int \ln x dx = x \ln x - x$	$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$
$\int x e^x dx = (x - 1)e^x$	$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$
$\int \tan x dx = \ln \sec x $	$\int \cot x dx = \ln \sin x $
$\int \sec x dx = \ln \sec x + \tan x $	$\int \csc x dx = -\ln \csc x + \cot x $
$\int \sec^2 x dx = \tan x$	$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4}$
$\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln \sec x + \tan x)$	
$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$
$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln x + \sqrt{x^2 \pm a^2} $
$\int \arctan x dx = x \arctan x - \frac{1}{2} \ln(1 + x^2)$	$\int \arcsin x dx = x \arcsin x + \sqrt{1 - x^2}$
$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$	
$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln x + \sqrt{x^2 \pm a^2} $	
$\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$	$\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

Integration by parts: $\int u dv = uv - \int v du$. Reduction: $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$.

SERIES

Geometric	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $ r < 1$
p -series	$\sum \frac{1}{n^p}$ converges iff $p > 1$; $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$
Finite sums	$\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$
Taylor	$f(x) = \sum \frac{f^{(n)}(a)}{n!} (x-a)^n$; $R_N = \frac{f^{(N+1)}(c)}{(N+1)!} (x-a)^{N+1}$

Test	Conclusion	Notes
Divergence	$a_n \not\rightarrow 0 \Rightarrow \text{diverges}$	Never proves convergence
Integral	$\sum f(n)$ and $\int_1^\infty f$ together	f positive, decreasing
Comparison	$0 \leq a_n \leq b_n, \sum b_n < \infty \Rightarrow \sum a_n < \infty$	Compare with p -series, geometric
Limit comparison	$\frac{a_n}{b_n} \rightarrow c \in (0, \infty)$: same behavior	Positive terms
Alternating	$b_n \downarrow 0 \Rightarrow \sum (-1)^n b_n$ converges	Error $\leq$ first omitted term
Ratio	$\left \frac{a_{n+1}}{a_n} \right \rightarrow L$: $L < 1$ converges, $L > 1$ diverges	Factorials, exponentials
Root	$ a_n ^{1/n} \rightarrow L$: same as ratio	n th powers

GREEK ALPHABET

α alpha	η eta	ν nu	τ tau	β beta	θ theta
ξ xi	υ epsilon	γ, Γ gamma	ι iota	o omicron	ϕ, φ, Φ phi
δ, Δ delta	κ kappa	π, Π pi	χ chi	ϵ, ε epsilon	λ, Λ lambda
ρ rho	ψ, Ψ psi	ζ zeta	μ mu	σ, Σ sigma	ω, Ω omega

APPENDIX C

NUMERICAL METHODS

Most equations cannot be solved in closed form, most integrals have no elementary antiderivative, and most differential equations have no formula for their solutions. This appendix collects the numerical methods of the text in four short Python programs, each followed by the output it actually produced. They use only the standard library. The theme is the same throughout: a method of *order* p has error proportional to h^p , so halving the step divides the error by 2^p , and the output tables show those ratios directly. Seeing the predicted ratio is the best evidence that a program is correct.

C.1 SOLVING EQUATIONS: BISECTION AND NEWTON'S METHOD

Bisection rests on the Intermediate Value Theorem: if $f(a)$ and $f(b)$ have opposite signs, a root lies in $[a, b]$; test the midpoint and keep the half on which the sign changes. Each step halves the interval, so n steps locate the root within $\frac{b-a}{2^n}$. It always works, but slowly: about 3.3 steps per decimal digit.

Newton's method (Chapter 6) replaces f by its tangent line: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. By Taylor's theorem with $N = 1$, if r is a simple root,

$$0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{1}{2}f''(c)(r - x_n)^2,$$

and dividing by $f'(x_n)$,

$$r - x_{n+1} = -\frac{f''(c)}{2f'(x_n)}(r - x_n)^2.$$

The error is squared at each step: *quadratic convergence*, which roughly doubles the number of correct digits per step once x_n is close to r .

```
# Bisection and Newton's method for f(x) = x^3 - 2x - 5 (Newton's own example,
1669)
f = lambda x: x**3 - 2*x - 5
df = lambda x: 3*x**2 - 2
```

```

a, b = 2.0, 3.0                                # f(2) = -1 < 0 < 16 = f(3)
print("bisection:  n    midpoint                interval width")
for n in range(1, 21):
    m = (a + b) / 2
    if f(a) * f(m) <= 0: b = m
    else: a = m
    if n % 5 == 0: print(f"{n:15d}    {m:.10f}    {b - a:.2e}")

x = 2.0
print("Newton:      n    x_n                    |f(x_n)|")
for n in range(1, 6):
    x = x - f(x) / df(x)
    print(f"{n:15d}    {x:.13f}    {abs(f(x)):.2e}")

```

OUTPUT

```

bisection:  n    midpoint                interval width
           5    2.0937500000    3.12e-02
          10    2.0947265625    9.77e-04
          15    2.0945739746    3.05e-05
          20    2.0945520401    9.54e-07

Newton:      n    x_n                    |f(x_n)|
           1    2.10000000000000    6.10e-02
           2    2.0945681211042    1.86e-04
           3    2.0945514816982    1.74e-09
           4    2.0945514815423    8.88e-16
           5    2.0945514815423    8.88e-16

```

Twenty bisection steps give six digits; three Newton steps give nine, and the fourth reaches the limit of double precision. The price is that Newton's method needs f' and a good starting point; started near a critical point of f it can be thrown far away. In practice the two are combined: bisect until the interval is small, then switch to Newton.

C.2 NUMERICAL INTEGRATION

The trapezoid rule and Simpson's rule of Chapter 9 have error bounds $\frac{K_2(b-a)^3}{12n^2}$ and $\frac{K_4(b-a)^5}{180n^4}$, where K_j bounds $|f^{(j)}|$. The program tests both on the integral $\int_0^1 e^{-x^2} dx$ of Example 13.15, whose exact value is available in terms of the error function.

```

# Trapezoid and Simpson rules for the integral of exp(-x^2) on [0, 1]
import math
f = lambda x: math.exp(-x * x)

```

```

exact = math.sqrt(math.pi) / 2 * math.erf(1)

def trap(n):
    h = 1 / n
    return h * (f(0) / 2 + sum(f(i * h) for i in range(1, n)) + f(1) / 2)

def simpson(n):
    h = 1 / n
    s = f(0) + f(1) + sum((4 if i % 2 else 2) * f(i * h) for i in range(1, n))
    return h / 3 * s

print(f"exact = {exact:.12f}")
print(" n   trapezoid error   ratio   Simpson error   ratio")
pt = ps = None
for n in [4, 8, 16, 32, 64]:
    et, es = trap(n) - exact, simpson(n) - exact
    rt = f"{pt / et:6.2f}" if pt else "      "
    rs = f"{ps / es:6.2f}" if ps else "      "
    print(f"{n:3d}   {et: .3e}      {rt}   {es: .3e}      {rs}")
    pt, ps = et, es

```

OUTPUT

```

exact = 0.746824132812
 n   trapezoid error   ratio   Simpson error   ratio
 4   -3.840e-03                3.125e-05
 8   -9.585e-04         4.01    1.988e-06    15.72
16   -2.395e-04         4.00    1.246e-07    15.95
32   -5.988e-05         4.00    7.795e-09    15.99
64   -1.497e-05         4.00    4.872e-10    16.00

```

The trapezoid ratios approach $4 = 2^2$ and the Simpson ratios $16 = 2^4$, as the bounds predict. The trapezoid errors are negative because e^{-x^2} is concave on most of $[0, 1]$ and the chords lie below the curve there. With $n = 64$, Simpson's rule is accurate to ten digits, using only 65 function values.

The ratios suggest an improvement. If $T_n \approx I + Ch^2$, then $T_{2n} \approx I + \frac{C}{4}h^2$, and eliminating C gives $I \approx \frac{4T_{2n} - T_n}{3}$, which is exactly Simpson's rule S_{2n} . Repeating this *Richardson extrapolation* on Simpson's rule gives *Romberg integration*, which is how many library routines work.

C.3 DIFFERENTIAL EQUATIONS: EULER TO RUNGE--KUTTA

Euler's method follows the slope at the start of each step and is first order (Example 10.2). Better methods sample the slope at several points of the step and average:

- The *improved Euler* (Heun) method takes a trial Euler step, evaluates the slope k_2 at its end, and uses the average $\frac{k_1+k_2}{2}$, as the trapezoid rule averages the endpoints. It is second order.
- The classical *fourth-order Runge–Kutta* method (RK4, 1901) uses four slopes,

$$\begin{aligned} k_1 &= F(x_n, y_n), & k_2 &= F\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\ k_3 &= F\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), & k_4 &= F(x_n + h, y_n + hk_3), \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

When F depends only on x , this is exactly Simpson's rule, and its error is fourth order.

```
# Euler, improved Euler (Heun), and fourth-order Runge-Kutta for y' = x + y, y
(0) = 1
import math
F = lambda x, y: x + y
exact = 2 * math.e - 2                                     # y = 2e^x - x - 1 at x = 1

def euler(h):
    x, y = 0.0, 1.0
    for _ in range(round(1 / h)):
        y += h * F(x, y); x += h
    return y

def heun(h):
    x, y = 0.0, 1.0
    for _ in range(round(1 / h)):
        k1 = F(x, y); k2 = F(x + h, y + h * k1)
        y += h * (k1 + k2) / 2; x += h
    return y

def rk4(h):
    x, y = 0.0, 1.0
    for _ in range(round(1 / h)):
        k1 = F(x, y)
        k2 = F(x + h / 2, y + h / 2 * k1)
        k3 = F(x + h / 2, y + h / 2 * k2)
```

```

        k4 = F(x + h, y + h * k3)
        y += h / 6 * (k1 + 2 * k2 + 2 * k3 + k4); x += h
    return y

print(f"exact y(1) = {exact:.12f}")
for name, method in [("Euler", euler), ("Heun", heun), ("RK4", rk4)]:
    prev = None
    for h in [0.1, 0.05, 0.025]:
        err = exact - method(h)
        ratio = f" ratio {prev / err:5.2f}" if prev else ""
        print(f"{name:6s} h = {h:<6} error = {err:.3e}{ratio}")
        prev = err

```

OUTPUT

```

exact y(1) = 3.436563656918
Euler h = 0.1      error = 2.491e-01
Euler h = 0.05     error = 1.300e-01  ratio  1.92
Euler h = 0.025    error = 6.644e-02  ratio  1.96
Heun  h = 0.1      error = 8.402e-03
Heun  h = 0.05     error = 2.182e-03  ratio  3.85
Heun  h = 0.025    error = 5.558e-04  ratio  3.93
RK4   h = 0.1      error = 4.169e-06
RK4   h = 0.05     error = 2.716e-07  ratio 15.35
RK4   h = 0.025    error = 1.733e-08  ratio 15.67

```

The ratios approach 2, 4, and 16. With $h = 0.1$, ten RK4 steps, forty evaluations of F in all, are about 65 times more accurate than ten thousand Euler steps. RK4, with automatic adjustment of the step size, is the standard tool for the differential equations of physics and engineering, including the systems of Volume IV.

C.4 ACCELERATING SERIES

The integral test (Theorem 12.12) does more than bound the remainder of a series: the bounds can be added to the partial sum as a correction. For $\sum \frac{1}{n^2}$, the remainder after N terms lies between $\frac{1}{N+1}$ and $\frac{1}{N}$, and the midpoint rule suggests the sharper estimate $\int_{N+1/2}^{\infty} x^{-2} dx = \frac{1}{N+1/2}$. For the alternating harmonic series, the sum lies between consecutive partial sums (Theorem 12.18), so their average should be closer than either.

```

# Accelerating slowly convergent series
import math

# (1) zeta(2) = pi^2/6: add the integral-test estimate of the tail, about 1/(N
+ 1/2)

```

```
target = math.pi ** 2 / 6
print("zeta(2):  N      s_N error          corrected error")
for N in [10, 100, 1000]:
    s = sum(1 / n ** 2 for n in range(1, N + 1))
    print(f"{N:14d}  {target - s:.3e}          {target - (s + 1 / (N + 0.5)):.3e}")

# (2) ln 2 = 1 - 1/2 + 1/3 - ...: average consecutive partial sums
print("ln 2:      N      s_N error          averaged error")
for N in [10, 100, 1000]:
    s = sum((-1) ** (n + 1) / n for n in range(1, N + 1))
    t = s + (-1) ** (N + 2) / (N + 1)          # s_{N+1}
    print(f"{N:14d}  {abs(math.log(2) - s):.3e}          {abs(math.log(2) - (s +
t) / 2):.3e}")
```

OUTPUT

zeta(2):	N	s_N error	corrected error
	10	9.517e-02	-7.176e-05
	100	9.950e-03	-8.209e-08
	1000	9.995e-04	-8.321e-11
ln 2:	N	s_N error	averaged error
	10	4.751e-02	2.058e-03
	100	4.975e-03	2.451e-05
	1000	4.998e-04	2.495e-07

The corrected sums gain several digits for free: one thousand terms of $\sum \frac{1}{n^2}$, corrected, give $\frac{\pi^2}{6}$ to ten places, where the uncorrected sum gives three. The corrected errors fall like N^{-3} and the averaged ones like N^{-2} , instead of N^{-1} . These are the simplest cases of general acceleration methods, the Euler–Maclaurin formula and Euler’s transformation of alternating series, with which Euler computed $\sum \frac{1}{n^2}$ accurately enough to recognize $\frac{\pi^2}{6}$ before he could prove it.

ANSWERS TO SELECTED EXERCISES

Answers are given for odd-numbered computational exercises. Every numerical answer here was checked by computer algebra. Proof exercises are not answered; check proofs against the definitions and theorems they rely on.

CHAPTER 1

FUNCTIONS AND THEIR GRAPHS

1.1 $(-3, \frac{5}{3}); (-\infty, 2] \cup [6, \infty)$.

1.3 $(1, 4); (-1, 0) \cup (1, \infty)$.

1.5 $[-2, 3) \cup (3, \infty)$.

1.7 $(-\infty, 0) \cup (4, \infty)$.

1.9 Even part $\frac{1}{1-x^2}$, odd part $\frac{x}{1-x^2}$.

1.11 $y = 3(x+2)^2 - 5$; vertex $(-2, -5)$.

1.13 $h = f \circ g \circ k$ with $k(x) = 2x + 1$,
 $g = \cos$, $f(u) = u^3$.

1.15 $f^{-1}(x) = \frac{1}{2}(1 + \ln(x-3))$, domain
 $(3, \infty)$.

1.17 1, 2, 3.

1.19 $x = \ln 2$ or $x = \ln 3$.

1.21 $x = \frac{\ln 2}{\ln(5/2)} \approx 0.7565$.

1.23 $\frac{3\pi}{4}, -\frac{\pi}{9}, 210^\circ$.

1.25 $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

1.27 $\frac{1}{\sqrt{1+x^2}}, \frac{1}{\sqrt{1-x^2}}$.

1.29 $\frac{5}{3}, \frac{3}{4}$.

CHAPTER 2

LIMITS AND CONTINUITY

2.1 The values 1.0517, 0.9516, 1.0050, 0.9950, 1.0005, 0.9995 suggest the limit 1.

2.3 Take $\delta = \epsilon/5$: then $|(4-5x)-9| = 5|x+1| < \epsilon$.

2.5 Take $\delta = \epsilon^{1/3}$: then $|x^3| = |x|^3 < \epsilon$.

2.7 $\frac{3}{2}$.

2.9 12.

2.11 $\frac{3}{7}$.

2.13 1.

2.15 $-\infty; +\infty$.

2.17 Vertical: $x = -1, x = 2$. Horizontal:
 $y = 2$.

2.19 $2; -\frac{3}{2}$.

2.21 1; -1.

2.23 $y = x$, since $f(x) = x + \frac{1-x}{x^2+1}$.

2.25 $a = \frac{4}{3}, b = \frac{2}{3}$.

2.27 $x \approx 0.74$ (more precisely 0.739085).

2.29 0; -1.

CHAPTER 3

THE DERIVATIVE

3.1 $f'(a) = 6a - 5$.

3.3 $f'(x) = \frac{1}{\sqrt{2x+1}}.$

3.5 $y = x - 2.$

3.7 $f(x) = x^{10}, a = 1: 10. f(x) = \tan x,$
 $a = \frac{\pi}{4}: 2.$

3.9 $2x + x^{-3/2}.$

3.11 $-\frac{2(x^2 + 3x + 1)}{(x^2 - 1)^2}.$

3.13 $x^2(x+3)e^x; \frac{(x-2)e^x}{x^3}.$

3.15 $\tan x + x \sec^2 x - \sec x \tan x.$

3.17 $-\frac{2}{(x+1)^3}.$

3.19 $\sin x; x^2 \sin x - 6x \cos x - 6 \sin x.$

3.21 $(2, \frac{1}{2})$ and $(-2, -\frac{1}{2}).$

3.23 At rest at $t = 2$ and $t = 4$; backward
on $(2, 4)$; distance $20 + 4 + 20 = 44$ m.

3.25 $C'(100) = \$9.00; C(101) - C(100) =$
 $\$9.0001.$

3.27 $a = 3, b = -4.$

3.29 $f'(x) = 2|x|$ for all x (including
 $f'(0) = 0$); f' is not differentiable at
0.

CHAPTER 4

THE CHAIN RULE AND ITS CONSEQUENCES

4.1 $42x^2(2x^3 - 5)^6; -\frac{6x}{(x^2 + 4)^4}.$

4.3 $6x \cos(3x^2 + 1); -8 \cos^3(2x) \sin(2x).$

4.5 $2x(1 - x^2)e^{-x^2}.$

4.7 0, since $\sec^2 u - \tan^2 u = 1$ is constant.

4.9 $\frac{2x}{(x^2 + 1) \ln 2}; 3^{\sin x} \ln 3 \cos x.$

4.11 $\arctan x.$

4.13 $y' = y \left(\frac{2}{x} + \frac{1}{2(x+1)} - \frac{3}{x-2} \right).$

4.15 $y' = -\frac{2x+y}{x+2y};$ at $(1, 2), y' = -\frac{4}{5}$ and
the tangent is $y = 2 - \frac{4}{5}(x - 1).$

4.17 $y' = \frac{y - e^{x+y}}{e^{x+y} - x}.$

4.19 Horizontal at $(1, 2)$ and $(-1, -2)$; ver-
tical at $(2, 1)$ and $(-2, -1).$

4.21 $\frac{1}{3a^2 + 3} \approx 0.2459,$ where $a \approx 0.5961$
is the real root of $a^3 + 3a = 2.$

4.23 The top slides down at 0.3 m/s.

4.25 480 km/h.

4.27 $L(x) = x - 1; \ln 1.05 \approx 0.05$ (true
value 0.04879).

4.29 Volume: $\pm 270 \text{ cm}^3$, relative error 1%.
Surface area: $\pm 36 \text{ cm}^2$, relative error
0.67%.

CHAPTER 5

THE MEAN VALUE THEOREM AND THE SHAPE OF GRAPHS

5.1 $f: -5, 1. g: 0$ (where g' is undefined),
 $\frac{8}{7}, 4.$

5.3 Maximum 6 at $x = 1$ and $x = 4$; mini-
mum -14 at $x = -1.$

5.5 Maximum $\frac{1}{2} + 2 \ln 2 \approx 1.886$ at $x = \frac{1}{2};$
minimum $2 - 2 \ln 2 \approx 0.614$ at $x = 2.$

5.7 $f(0) = f(3) = 2; c = \frac{1+\sqrt{19}}{3} \approx 1.786.$

5.9 $c = e - 1.$

5.11 Decreasing on $(-\infty, -1)$ and $(0, 1),$
increasing on $(-1, 0)$ and $(1, \infty)$; local
minima $f(\pm 1) = 2$, local maximum
 $f(0) = 3.$

5.13 Increasing on $(0, 2)$; decreasing on
 $(-\infty, 0)$ and $(2, \infty).$

5.15 Concave up on $(-1, 1),$ down on
 $|x| > 1$; inflection points $(\pm 1, \ln 2).$

5.17 At $x = 4: f''(4) = 320 > 0$, local

minimum -256 . At $x = 0$ the test fails; the first derivative test gives a local maximum $f(0) = 0$.

5.19 Even; intercepts $(0, \frac{4}{9})$, $(\pm 2, 0)$; asymptotes $x = \pm 3$, $y = 1$; increasing for $x < 0$, decreasing for $x > 0$ (away from ± 3); local maximum $(0, \frac{4}{9})$; concave up for $|x| > 3$, down for $|x| < 3$; no inflection points.

5.21 Local minimum at $x = \frac{\pi}{3}$, value $\frac{\pi}{3} - \sqrt{3} \approx -0.685$; local maximum at $x = \frac{5\pi}{3}$, value $\frac{5\pi}{3} + \sqrt{3} \approx 6.968$; inflection at (π, π) .

5.23 Maximum 0 at $x = 0$; concave down throughout; vertical asymptotes $x = \pm \frac{\pi}{2}$.

5.25 $a = 0$, $b = -3$: $f(x) = x^3 - 3x$.

5.27 Local maximum $f(-1) = -2$, local minimum $f(1) = 2$; they lie on different branches, separated by the asymptote $x = 0$.

5.29 $f(2) \leq 7$.

CHAPTER 6

APPLICATIONS OF THE DERIVATIVE

6.1 $\frac{1}{3}$.

6.3 0; 0.

6.5 0; 0.

6.7 0.

6.9 e^2 ; $e^{-1/2}$.

6.11 10 and 10.

6.13 150 m perpendicular to the river by 300 m along it; area 45 000 m².

6.15 $(-\frac{6}{5}, \frac{3}{5})$.

6.17 Height $\frac{4R}{3}$; volume $\frac{32\pi R^3}{81}$.

6.19 $q = 2500$ units at $p = \$35$.

6.21 $\sqrt[3]{5} \approx 1.709976$.

6.23 $x \approx 0.567143$.

6.25 $2x^3 - 4x^2 + 3x + C$; $2 \ln |x| + 3e^x - \tan x + C$.

6.27 $f(x) = 4 \sin x + 3 \cos x - 1$.

6.29 After about 9.09 s, at about 94.0 m/s.

CHAPTER 7

THE DEFINITE INTEGRAL

7.1 $R_4 = 0.950$, $L_4 \approx 1.283$; since $\frac{1}{x}$ decreases, L_4 overestimates and R_4 underestimates ($\ln 3 \approx 1.099$).

7.3 $\int_1^4 \sqrt{x} dx = \frac{14}{3}$.

7.5 $\frac{19}{6}$.

7.7 $\frac{8}{3}$.

7.9 $g'(x) = \cos(x^3)$; $h'(x) = \frac{2x}{\ln x^2} - \frac{1}{\ln x} = \frac{x-1}{\ln x}$.

7.11 Displacement 4 m; distance 12 m.

7.13 $\frac{1}{20}(2x^2 + 3)^5 + C$.

7.15 $\frac{2}{3}(1 + e^x)^{3/2} + C$.

7.17 $\frac{1}{2} \arctan(x^2) + C$.

7.19 $\frac{1}{2}(1 - \frac{1}{e})$.

7.21 $\frac{1}{3}$.

7.23 $\frac{\pi}{6}$; $\frac{\pi}{18}$.

7.25 $\frac{4}{15}$.

7.27 $f(x) = e^x$.

7.29 $(-1, 1)$.

CHAPTER 8

APPLICATIONS OF INTEGRATION

8.1 $\frac{1}{2}$.

8.3 9.

8.5 8π .

8.7 $\frac{64\pi}{15}$.

8.9 $\frac{1}{3}\pi r^2 h$.

8.11 $\frac{\pi^2}{2}$.

8.13 $\frac{17}{12}$.

8.15 $L = \int_0^1 \sqrt{1+4x^2} dx$; $T_4 \approx 1.4883$
(true value 1.4789).

8.17 $\pi r \ell$.

8.19 $490\,000\pi \approx 1.54 \times 10^6$ J.

8.21 313 600 N.

8.23 $(\frac{\pi}{2}, \frac{\pi}{8})$.

8.25 8π (area 2, centroid at $\bar{x} = 2$).

8.27 $e^{-1.5} \approx 0.223$; median $1000 \ln 2 \approx 693$ hours.

8.29 $6\pi^2$.

CHAPTER 9

TECHNIQUES OF INTEGRATION

9.1 $\frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + C$; $-(x^2 + 2x + 2)e^{-x} + C$.

9.3 $\frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$.

9.5 $-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$; $\frac{x}{8} - \frac{\sin 4x}{32} + C$.

9.7 $-\frac{\cos 7x}{14} - \frac{\cos 3x}{6} + C$.

9.9 $\frac{\sqrt{2}}{8}$.

9.11 $2 \ln |x - 1| + \frac{1}{2} \ln |2x + 1| + C$.

9.13 $\ln |x + 1| - \ln |x| - \frac{1}{x} + C$.

9.15 $\frac{\pi}{8} - \frac{1}{4}$.

9.17 $T_4 \approx 0.742984$, $M_4 \approx 0.748747$, $S_4 \approx 0.746855$.

9.19 $\frac{1}{2}$; 1; diverges.

9.21 1; diverges.

9.23 Converges; diverges; converges.

9.25 24; $\frac{\sqrt{\pi}}{2}$; $\frac{15}{8}$.

9.27 $\frac{2}{15}(x+1)^{3/2}(3x-2) + C$.

9.29 $\frac{\pi}{4}$.

CHAPTER 10

DIFFERENTIAL EQUATIONS

10.1 $y = \frac{\ln x + 2}{x}$.

10.3 The line $y = x + 1$ is a solution;
through $(0, 0)$: $y = x + 1 - e^x$;
through $(0, 2)$: $y = x + 1 + e^x$.

10.5 $y(0.5) \approx 0.0300$.

10.7 $y = \tan(\arctan x + \frac{\pi}{4}) = \frac{1+x}{1-x}$.

10.9 $y = 2e^{\sin t}$.

10.11 About 15.5 years.

10.13 35°C .

10.15 Lines $y = kx$ through the origin; hyperbolas $x^2 - y^2 = C$.

10.17 $y = \frac{x^3 - x}{2}$.

10.19 $y = \frac{C - \cos t}{t^3}$.

10.21 $10e^{-1.2} \approx 3.01$ kg.

10.23 $y = -1$ unstable, $y = 0$ stable, $y = 1$ unstable.

10.25 $(0, 0)$ and $(500, 50)$; $0.1 \ln W - 0.002W + 0.2 \ln R - 0.0004R = K$.

10.27 $y^{-2} = x + \frac{1}{2} + Ce^{2x}$.

10.29 $y^2 = x^2(2 \ln x + C)$.

CHAPTER 11

PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 The line $y = \frac{2x+17}{3}$.

11.3 $x^2 - y^2 = 1$, $x \geq 1$: the right branch, traced upward.

11.5 $x = 1 + 3t$, $y = 2 - 5t$, $0 \leq t \leq 1$.

11.7 $y = \frac{x}{4} - \frac{1}{4}$.

11.9 Horizontal at $(0, -4)$; vertical at

$(-2, -3)$ and $(2, -3)$.

11.11 $-\frac{1}{(1-\cos t)^2} < 0$.

11.13 $2 - \frac{4}{e}$.

11.15 $\frac{13\sqrt{13}-8}{27}$.

11.17 12.

11.19 $\frac{24\pi(1+\sqrt{2})}{5}$.

11.21 $(3\sqrt{2}, \frac{3\pi}{4}); (2, \frac{5\pi}{3})$.

11.23 $r^2 \cos 2\theta = 1; r = \tan \theta \sec \theta$.

11.25 $-\frac{2}{\pi}$.

11.27 $\frac{\pi}{12}$.

11.29 $\pi - \frac{3\sqrt{3}}{2}$.

11.31 Ellipse; $e = \frac{1}{2}$, $r_{\min} = 2$, $r_{\max} = 6$,
 $a = 4$.

CHAPTER 12

INFINITE SEQUENCES AND SERIES

12.1 $\frac{3}{5}$.

12.3 e^2 .

12.5 Diverges: the terms approach 1 and -1 alternately.

12.7 6.

12.9 3.

12.11 $\frac{41}{333}$.

12.13 $s_N = -\ln(N+1) \rightarrow -\infty$.

12.15 Diverges: $a_n \rightarrow 1$.

12.17 Converges; $\int_1^\infty xe^{-x^2} dx = \frac{1}{2e}$.

12.19 $s_5 \approx 1.08035$; $1.08190 \leq s \leq 1.08302$.

12.21 Converges; compare with $\frac{1}{n^{3/2}}$.

12.23 Converges; limit comparison with $(\frac{2}{3})^n$.

12.25 Conditionally.

12.27 Absolutely.

12.29 Converges; ratio $\rightarrow \frac{1}{3}$.

12.31 Converges; ratio $\rightarrow \frac{4}{5}$.

CHAPTER 13

POWER SERIES AND TAYLOR SERIES

13.1 $[-1, 1]$.

13.3 $R = 2$; $[-3, 1)$.

13.5 All real x .

13.7 $R = 0$.

13.9 $\sum (-3)^n x^n$; $R = \frac{1}{3}$.

13.11 $-\sum_{n \geq 1} \frac{x^{2n}}{n}$; $R = 1$.

13.13 $\sum_{n \geq 0} \frac{(-1)^n x^{4n+3}}{2n+1}$; $R = 1$.

13.15 $\ln \frac{3}{2}$.

13.17 $\sum_{n \geq 0} \frac{(-1)^{n+1} (x-\pi/2)^{2n+1}}{(2n+1)!}$.

13.19 $\sum_{n \geq 0} \frac{(-1)^n x^{n+2}}{n!}$.

13.21 $1 + \frac{x^2}{2} + \frac{5x^4}{24}$.

13.23 $T_3 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$;
 $T_3(1.2) \approx 0.182667$; error at most 0.0004.

13.25 1.0196.

13.27 $-\frac{1}{24}$.

13.29 0.310.

13.31 $\cos \frac{\pi}{2} = 0$.

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