



FORGED MATHEMATICS SERIES · VOLUME II

PRECALCULUS & TRIGONOMETRY

FUNCTIONS · TRIGONOMETRY · ANALYTIC GEOMETRY

MATHEMATICS

FORGED OFFICIAL TEXTBOOK

A thirteen-chapter text on the mathematics calculus is built on: functions and their graphs; polynomial, rational, exponential, and logarithmic functions; trigonometry; complex numbers, vectors, and conic sections; systems; and sequences.

First edition · Theorems, proofs, worked examples, and verified exercises



FORGED MATHEMATICS SERIES • VOLUME II

PRECALCULUS & TRIGONOMETRY

FUNCTIONS • TRIGONOMETRY • ANALYTIC GEOMETRY

Omnistrata, LLC

First Edition

ForgEd • forGEDlearn.com
Learn skills that matter.

Precalculus & Trigonometry

ForgEd Mathematics Series, Volume II

First Edition

Copyright © 2026 Omnistrata, LLC. All rights reserved.

Published by Omnistrata, LLC for Forged (forgedlearn.com). No part of this book may be reproduced or transmitted in any form without written permission of the publisher, except for brief quotations in reviews.

The Forged Mathematics Series

Volume I Algebra

Volume II Precalculus & Trigonometry

Volume III Calculus I & II

Volume IV Multivariable Calculus & Quantum Mechanics

Typeset in TeX Gyre Pagella and Cinzel.

PREFACE

This book covers precalculus and trigonometry: the algebra, functions, and geometry that calculus takes for granted. It is written for a reader who has finished algebra (Volume I of this series) and wants to understand these topics well enough that calculus feels like a continuation rather than a new language. Most students who struggle in calculus are not struggling with limits or derivatives. They are struggling with the algebra and trigonometry underneath, and this book is designed to remove that obstacle.

HOW THE BOOK IS ORGANIZED

Part I (Chapters 1–4) develops functions: the real numbers and inequalities, functions and their graphs, polynomial and rational functions, and exponential and logarithmic functions. **Part II** (Chapters 5–8) is a complete course in trigonometry: angles and the circular functions, their graphs and inverses, identities and equations, and the solution of triangles, including the ambiguous case. **Part III** (Chapters 9–13) treats complex numbers and De Moivre’s Theorem, polar coordinates and vectors, conic sections (including rotated and polar forms), systems and matrices, and sequences, series, induction, and the binomial theorem.

HOW TO READ THIS BOOK

Each chapter opens with its learning objectives. Boxed material comes in four kinds.

Definition 0.1 (Definitions, theorems, propositions). Shaded boxes contain the statements you should be able to reproduce from memory. Theorems carry a gold rule.

Example 0.2 (Examples). Worked problems. Cover the solution and attempt the problem before reading it.

COMMON ERROR. Black-ruled boxes describe the specific mistakes students make most often, and how to detect them.

LOOKING AHEAD. Dark boxes connect the current material to where it is used later, in calculus (Volume III) or beyond (Volume IV).

Theorems, definitions, and examples share one numbering sequence within each chapter.

EXERCISES AND ANSWERS

Exercises are grouped as *Computation*, *Concepts and Proof*, and *Challenge*. Answers to odd-numbered exercises with a short final answer appear at the back of the book. Every numerical answer in the examples and in the answer section was checked by computer algebra before publication; if your result disagrees, find the step where it diverges.

NOTATION

Angles are measured in radians unless a degree sign appears. The inverse trigonometric functions are written $\arcsin$, $\arccos$, $\arctan$; the notation $\sin^{-1} x$ means the same thing as $\arcsin x$ and never $1/\sin x$, while $\sin^2 x$ means $(\sin x)^2$. An interval $[a, b]$ is closed and (a, b) is open.

CONTENTS

Preface	iv
Part I Functions	1
1 Numbers, Equations, and Inequalities	2
1.1 The Real Number Line	2
1.2 Equations	4
1.3 Inequalities	5
1.4 The Coordinate Plane	6
1.5 Worked Problems	8
Skill Drills	10
Exercises	11
2 Functions and Their Graphs	14
2.1 Functions	14
2.2 Graphs of Functions	16
2.3 Transformations	17
2.4 Combining Functions	18
2.5 Inverse Functions	18
2.6 Worked Problems	19
Skill Drills	21
Exercises	22
3 Polynomial and Rational Functions	26
3.1 Quadratic Functions	26

3.2	Polynomial Functions and Their Graphs	27
3.3	Dividing Polynomials	28
3.4	Zeros of Polynomials	29
3.5	Rational Functions	30
3.6	Worked Problems	31
	Skill Drills	34
	Exercises	34
4	Exponential and Logarithmic Functions	38
4.1	Exponential Functions	39
4.2	Logarithmic Functions	40
4.3	Laws of Logarithms	41
4.4	Exponential and Logarithmic Equations	42
4.5	Modeling	42
4.6	Further Models	43
4.7	Worked Problems	45
	Skill Drills	47
	Exercises	48
Part II	Trigonometry	51
5	Angles and the Trigonometric Functions	52
5.1	Angles and Their Measure	52
5.2	The Unit Circle	54
5.3	Fundamental Identities	55
5.4	Right Triangle Trigonometry	56
5.5	Worked Problems	57
	Skill Drills	58
	Exercises	59
6	Trigonometric Graphs and Inverse Functions	62
6.1	Graphs of Sine and Cosine	62

6.2	Graphs of the Other Trigonometric Functions	64
6.3	Inverse Trigonometric Functions	65
6.4	More on Trigonometric Graphs	66
6.5	Worked Problems	68
	Skill Drills	70
	Exercises	71
7	Trigonometric Identities and Equations	74
7.1	Verifying Identities	74
7.2	Addition and Subtraction Formulas	75
7.3	Double-Angle, Half-Angle, and Product–Sum Formulas	76
7.4	Trigonometric Equations	78
7.5	Worked Problems	79
	Skill Drills	81
	Exercises	82
8	Solving Triangles	85
8.1	The Law of Sines	85
8.2	The Law of Cosines	87
8.3	Area of a Triangle	87
8.4	Applications	88
8.5	Worked Problems	89
	Skill Drills	91
	Exercises	91
Part III	Analytic Geometry and Algebra	95
9	Complex Numbers	96
9.1	Arithmetic of Complex Numbers	96
9.2	The Complex Plane	97
9.3	Trigonometric Form	98
9.4	De Moivre’s Theorem	98

9.5	Roots of Complex Numbers	99
9.6	Worked Problems	100
	Skill Drills	102
	Exercises	103
10	Polar Coordinates, Parametric Equations, and Vectors	106
10.1	Polar Coordinates	106
10.2	Parametric Equations	108
10.3	Vectors in the Plane	109
10.4	The Dot Product	109
10.5	Worked Problems	110
	Skill Drills	112
	Exercises	113
11	Conic Sections	116
11.1	Parabolas	116
11.2	Ellipses	117
11.3	Hyperbolas	118
11.4	Shifted Conics	119
11.5	Rotation of Axes	119
11.6	Polar Equations of Conics	120
11.7	Worked Problems	121
	Skill Drills	123
	Exercises	124
12	Systems of Equations and Matrices	127
12.1	Linear Systems	127
12.2	Matrix Algebra	129
12.3	Inverses and Determinants	129
12.4	Nonlinear Systems and Partial Fractions	130
12.5	Systems of Inequalities and Linear Programming	131
12.6	Worked Problems	134
	Skill Drills	137

Exercises	138
13 Sequences, Series, and Induction	141
13.1 Sequences and Summation Notation	141
13.2 Arithmetic Sequences	142
13.3 Geometric Sequences and Series	143
13.4 Mathematical Induction	144
13.5 The Binomial Theorem	145
13.6 Worked Problems	146
Skill Drills	148
Exercises	149
A Formula Reference	152
A.1 Algebra	152
A.2 Geometry	153
A.3 Trigonometry	153
B The Unit Circle	155
Answers to Selected Exercises	157



PART I

FUNCTIONS

Numbers, equations, and inequalities; functions and their graphs; polynomial, rational, exponential, and logarithmic functions.

CHAPTER 1

NUMBERS, EQUATIONS, AND INEQUALITIES

LEARNING OBJECTIVES

- Work with the real number line, intervals, and absolute value, and prove that $\sqrt{2}$ is irrational.
- Solve linear, quadratic, radical, and absolute value equations, and detect extraneous solutions.
- Derive the quadratic formula and use the discriminant to classify roots.
- Solve linear, absolute value, polynomial, and rational inequalities with sign charts.
- Use the distance and midpoint formulas, and write equations of circles and lines.

Every function in this book takes real numbers as inputs and produces real numbers as outputs, and almost every problem eventually comes down to solving an equation or an inequality. This chapter reviews those foundations with more care than a first algebra course usually gives them, because the habits formed here, especially checking for extraneous solutions and reading sign charts, are used constantly in calculus.

1.1 THE REAL NUMBER LINE

The number systems are nested: the *natural numbers* $\mathbb{N} = \{1, 2, 3, \dots\}$, the *integers* $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the *rational numbers* $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$, and the *real numbers* $\mathbb{R}$, which correspond to the points of a line. A real number that is not rational is *irrational*. Rational numbers have decimal expansions that terminate or repeat; irrational numbers have expansions that do neither.

Theorem 1.1 ($\sqrt{2}$ is irrational). *There is no rational number whose square is 2.*

Proof. Suppose $\sqrt{2} = p/q$ with integers p, q having no common factor. Then $p^2 = 2q^2$, so p^2 is even, and therefore p is even (the square of an odd number is odd). Write $p = 2k$. Then $4k^2 = 2q^2$, so $q^2 = 2k^2$ is even and q is even. Now p and q share the factor 2, a contradiction. ■

This proof, known to the Pythagoreans, is the model for every *proof by contradiction*: assume the opposite, and deduce something impossible.

1.1.1 INTERVALS

Notation	Set	Description
(a, b)	$\{x : a < x < b\}$	open interval
$[a, b]$	$\{x : a \leq x \leq b\}$	closed interval
$[a, b)$	$\{x : a \leq x < b\}$	half-open
(a, ∞)	$\{x : x > a\}$	unbounded
$(-\infty, b]$	$\{x : x \leq b\}$	unbounded

The symbol ∞ is not a number; a parenthesis always appears next to it. The *union* $A \cup B$ is the set of numbers in A or B (or both), and the *intersection* $A \cap B$ is the set in both.

1.1.2 ABSOLUTE VALUE

Definition 1.2 (Absolute value).

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

Geometrically, $|a|$ is the distance from a to 0, and $|a - b|$ is the distance between a and b on the number line.

Theorem 1.3 (Properties of absolute value). *For all real a, b : (1) $|a| \geq 0$; (2) $|ab| = |a||b|$; (3) $|a/b| = |a|/|b|$ if $b \neq 0$; (4) $\sqrt{a^2} = |a|$; (5) the triangle inequality $|a + b| \leq |a| + |b|$.*

Proof of the triangle inequality. Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, adding gives $-(|a| + |b|) \leq a + b \leq |a| + |b|$, which says exactly $|a + b| \leq |a| + |b|$. ■

COMMON ERROR ($\sqrt{a^2} = a$). $\sqrt{a^2} = |a|$, not a . With $a = -3$: $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$. This error reappears in calculus whenever a square root is simplified for negative inputs.

1.2 EQUATIONS

To *solve* an equation is to find every value of the variable that makes it true. Two equations are *equivalent* if they have the same solutions. Adding the same quantity to both sides, or multiplying both sides by the same nonzero quantity, produces an equivalent equation. Squaring both sides, or multiplying by an expression that might be zero, does *not*, and may introduce *extraneous solutions* that must be removed by checking.

1.2.1 QUADRATIC EQUATIONS

Theorem 1.4 (Quadratic formula). *The solutions of $ax^2 + bx + c = 0$ with $a \neq 0$ are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant $D = b^2 - 4ac$ determines the type: two real solutions if $D > 0$, one (repeated) if $D = 0$, and none that are real if $D < 0$ (two complex solutions, Chapter 9).

Proof. Divide by a and complete the square: $x^2 + \frac{b}{a}x = -\frac{c}{a}$, so

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}, \quad \text{that is} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots, $x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, provided $b^2 - 4ac \geq 0$. ■

Example 1.5 (Three methods). (a) $x^2 - 5x - 14 = 0$ factors as $(x - 7)(x + 2) = 0$: $x = 7$ or $x = -2$. (b) $2x^2 + 4x - 1 = 0$: $x = \frac{-4 \pm \sqrt{16 + 8}}{4} = \frac{-2 \pm \sqrt{6}}{2}$. (c) $x^2 + 2x + 5 = 0$: $D = 4 - 20 < 0$, so there are no real solutions.

1.2.2 EQUATIONS THAT ARE NOT QUADRATIC

Example 1.6 (Quadratic in form). Solve $x^4 - 5x^2 + 4 = 0$ and $x^{2/3} - x^{1/3} - 6 = 0$.

Solution. Let $u = x^2$: $u^2 - 5u + 4 = 0$, $u = 1$ or 4 , so $x = \pm 1, \pm 2$. For the second, let $u = x^{1/3}$: $u^2 - u - 6 = 0$, $u = 3$ or -2 , so $x = 27$ or $x = -8$.

Example 1.7 (A radical equation). Solve $\sqrt{2x+3} = x$.

Solution. Squaring, $2x + 3 = x^2$, so $x^2 - 2x - 3 = 0$ and $x = 3$ or $x = -1$. Check: $\sqrt{9} = 3$ works; $\sqrt{1} = 1 \neq -1$ fails. The only solution is $x = 3$. The extraneous root -1 is a solution of $-\sqrt{2x+3} = x$, which has the same square.

Example 1.8 (Absolute value equations). Solve $|2x - 5| = 3$ and $|x - 1| = |2x + 3|$.

Solution. $|2x - 5| = 3$ means $2x - 5 = 3$ or $2x - 5 = -3$: $x = 4$ or $x = 1$. For the second, $|u| = |v|$ means $u = \pm v$: $x - 1 = 2x + 3$ gives $x = -4$, and $x - 1 = -(2x + 3)$ gives $x = -\frac{2}{3}$.

COMMON ERROR (Dividing by the variable). Solving $x^2 = 3x$ by dividing by x gives $x = 3$ and loses $x = 0$. Factor instead: $x(x - 3) = 0$. Never divide by an expression that could be zero.

1.3 INEQUALITIES

Inequalities are solved like equations, with one difference: *multiplying or dividing by a negative number reverses the inequality*. ($2 < 3$, but $-2 > -3$.)

Theorem 1.9 (Absolute value inequalities). For $c > 0$: $|u| < c \iff -c < u < c$, and $|u| > c \iff u < -c$ or $u > c$.

Example 1.10 (Absolute value inequalities). $|x - 4| < 0.5$ means $3.5 < x < 4.5$: the numbers within 0.5 of 4. And $|3x + 2| \geq 4$ means $3x + 2 \leq -4$ or $3x + 2 \geq 4$, that is $x \in (-\infty, -2] \cup [\frac{2}{3}, \infty)$.

1.3.1 POLYNOMIAL AND RATIONAL INEQUALITIES

A polynomial or rational expression can change sign only where it is zero or undefined. These *critical values* divide the line into intervals on which the sign is constant, and one test point per interval determines it.

Example 1.11 (A sign chart). Solve $x^3 - x^2 - 6x > 0$.

Solution. Factor: $x(x - 3)(x + 2) > 0$. Critical values $-2, 0, 3$.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 3)$	$(3, \infty)$
x	—	—	+	+
$x - 3$	—	—	—	+
$x + 2$	—	+	+	+
Product	—	+	—	+

The solution is $(-2, 0) \cup (3, \infty)$.

Example 1.12 (A rational inequality). Solve $\frac{x+1}{x-2} \leq 3$.

Solution. Do not multiply by $x - 2$, whose sign is unknown. Move everything to one side: $\frac{x+1}{x-2} - 3 = \frac{x+1-3(x-2)}{x-2} = \frac{7-2x}{x-2} \leq 0$. Critical values 2 (undefined) and $\frac{7}{2}$ (zero). Testing: negative on $(-\infty, 2)$, positive on $(2, \frac{7}{2})$, negative on $(\frac{7}{2}, \infty)$. Including the zero but not the point where it is undefined: $(-\infty, 2) \cup [\frac{7}{2}, \infty)$.

COMMON ERROR (Multiplying an inequality by an expression). Multiplying $\frac{x+1}{x-2} \leq 3$ by $x - 2$ gives $x + 1 \leq 3x - 6$, so $x \geq \frac{7}{2}$. That loses the whole interval $(-\infty, 2)$, where $x - 2 < 0$ and the inequality should have reversed.

1.4 THE COORDINATE PLANE

Theorem 1.13 (Distance and midpoint). The distance between (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, and their midpoint is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The distance formula is the Pythagorean theorem applied to the right triangle with legs $|x_2 - x_1|$ and $|y_2 - y_1|$. A circle with center (h, k) and radius r is the set

of points at distance r from the center, so its equation is

$$(x - h)^2 + (y - k)^2 = r^2.$$

Example 1.14 (Completing the square). Identify $x^2 + y^2 + 6x - 4y - 12 = 0$.

Solution. Group and complete squares: $(x^2 + 6x + 9) + (y^2 - 4y + 4) = 12 + 9 + 4$, so $(x + 3)^2 + (y - 2)^2 = 25$: the circle with center $(-3, 2)$ and radius 5.

1.4.1 LINES

The *slope* of the line through (x_1, y_1) and (x_2, y_2) , $x_1 \neq x_2$, is $m = \frac{y_2 - y_1}{x_2 - x_1}$: the rise per unit run. Lines can be written in *point-slope form* $y - y_1 = m(x - x_1)$, *slope-intercept form* $y = mx + b$, or *general form* $Ax + By + C = 0$. Vertical lines $x = a$ have undefined slope.

Theorem 1.15 (Parallel and perpendicular lines). *Two nonvertical lines are parallel if and only if their slopes are equal, and perpendicular if and only if their slopes satisfy $m_1 m_2 = -1$.*

Proof of the perpendicular condition. Translate so that both lines pass through the origin: $y = m_1 x$ and $y = m_2 x$. They contain the points $A(1, m_1)$ and $B(1, m_2)$. The lines are perpendicular exactly when triangle OAB has a right angle at O , that is (by Pythagoras and its converse) when $|OA|^2 + |OB|^2 = |AB|^2$: $(1 + m_1^2) + (1 + m_2^2) = (m_1 - m_2)^2$. Expanding, $2 = -2m_1 m_2$, so $m_1 m_2 = -1$. ■

Example 1.16 (A perpendicular bisector). Find the perpendicular bisector of the segment from $A(1, 2)$ to $B(5, -4)$.

Solution. Midpoint $(3, -1)$; slope of AB is $\frac{-6}{4} = -\frac{3}{2}$; perpendicular slope $\frac{2}{3}$. The bisector is $y + 1 = \frac{2}{3}(x - 3)$, or $2x - 3y = 9$. Every point on it is equidistant from A and B .

LOOKING AHEAD. Absolute value inequalities are the language of limits: the statement $|x - a| < \delta$ means “ x is within δ of a ,” and the definition of $\lim_{x \rightarrow a} f(x) = L$ in Volume III is a pair of such inequalities. Sign charts reappear in calculus to find where a function is increasing (the sign of its derivative) and where its graph bends upward (the sign of its second deriva-

tive).

1.5 WORKED PROBLEMS

The problems below combine several ideas from the chapter. Each solution shows the full reasoning, including the checks that catch errors. Try each one before reading the solution.

PROBLEM 1.1 Two absolute values

Solve $|x - 3| + |2x + 1| = 10$.

Solution. Each absolute value changes formula where its inside is zero: at $x = 3$ and $x = -\frac{1}{2}$. These split the line into three intervals; on each, remove the bars with the correct signs.

Case $x < -\frac{1}{2}$: both insides are negative, so $-(x - 3) - (2x + 1) = 10$, that is $-3x + 2 = 10$ and $x = -\frac{8}{3}$. This lies in the interval, so it counts.

Case $-\frac{1}{2} \leq x < 3$: $-(x - 3) + (2x + 1) = 10$, so $x + 4 = 10$ and $x = 6$. This is *not* in $[-\frac{1}{2}, 3)$: reject it.

Case $x \geq 3$: $(x - 3) + (2x + 1) = 10$, so $3x - 2 = 10$ and $x = 4$. This is in the interval.

The solutions are $x = -\frac{8}{3}$ and $x = 4$. Check: $|- \frac{17}{3}| + |-\frac{13}{3}| = \frac{30}{3} = 10$, and $1 + 9 = 10$.

PROBLEM 1.2 Two radicals

Solve $\sqrt{3x + 1} - \sqrt{x - 1} = 2$.

Solution. Isolate one radical before squaring: $\sqrt{3x + 1} = 2 + \sqrt{x - 1}$. Squaring, $3x + 1 = 4 + 4\sqrt{x - 1} + x - 1$, so $2x - 2 = 4\sqrt{x - 1}$, that is $x - 1 = 2\sqrt{x - 1}$. Let $u = \sqrt{x - 1} \geq 0$: then $u^2 = 2u$, so $u = 0$ or $u = 2$, giving $x = 1$ or $x = 5$.

Both squarings could have introduced extraneous roots, so check in the original: $x = 1$ gives $\sqrt{4} - 0 = 2$; $x = 5$ gives $\sqrt{16} - \sqrt{4} = 2$. Both are solutions. Note that dividing $x - 1 = 2\sqrt{x - 1}$ by $\sqrt{x - 1}$ would have lost $x = 1$.

PROBLEM 1.3 Comparing two fractions

Solve $\frac{x + 3}{x - 1} \geq \frac{x - 2}{x + 2}$.

Solution. Move everything to one side and combine over a common denominator:

$$\begin{aligned}\frac{x+3}{x-1} - \frac{x-2}{x+2} &= \frac{(x+3)(x+2) - (x-2)(x-1)}{(x-1)(x+2)} \\ &= \frac{(x^2 + 5x + 6) - (x^2 - 3x + 2)}{(x-1)(x+2)} = \frac{4(2x+1)}{(x-1)(x+2)}.\end{aligned}$$

Critical values: -2 and 1 (undefined) and $-\frac{1}{2}$ (zero). Signs on $(-\infty, -2)$, $(-2, -\frac{1}{2})$, $(-\frac{1}{2}, 1)$, $(1, \infty)$: $-, +, -, +$. We want ≥ 0 , including the zero but not the undefined points:

$$(-2, -\frac{1}{2}] \cup (1, \infty).$$

PROBLEM 1.4 A parameter in a quadratic

For which values of k does $kx^2 + (k+3)x + k = 0$ have two distinct real solutions?

Solution. First, $k \neq 0$, or the equation is not quadratic (with $k = 0$ it is $3x = 0$, one solution). For a quadratic, two distinct real roots require $D > 0$:

$$D = (k+3)^2 - 4k^2 = -3k^2 + 6k + 9 = -3(k-3)(k+1).$$

$D > 0$ exactly when $(k-3)(k+1) < 0$, that is $-1 < k < 3$. Excluding $k = 0$: the answer is $(-1, 0) \cup (0, 3)$.

PROBLEM 1.5 A circle through three points

Find the circle through $(0,0)$, $(4,0)$, and $(0,6)$.

Solution. The center (h,k) is equidistant from all three points. Equidistant from $(0,0)$ and $(4,0)$ means it is on the perpendicular bisector $x = 2$; from $(0,0)$ and $(0,6)$, on $y = 3$. So the center is $(2,3)$, and the radius is its distance to the origin, $\sqrt{13}$:

$$(x-2)^2 + (y-3)^2 = 13.$$

The three points form a right angle at the origin, so the hypotenuse from $(4,0)$ to $(0,6)$ is a diameter, and its midpoint $(2,3)$ is the center: a second way to the same answer.

PROBLEM 1.6 A tangent line

Find the tangent line to the circle $x^2 + y^2 = 25$ at $(3, 4)$, and the distance from the origin to that line.

Solution. A tangent to a circle is perpendicular to the radius at the point of contact. The radius to $(3, 4)$ has slope $\frac{4}{3}$, so the tangent has slope $-\frac{3}{4}$: $y - 4 = -\frac{3}{4}(x - 3)$, or $3x + 4y = 25$. Its distance from the origin is $\frac{|0+0-25|}{\sqrt{9+16}} = 5$, the radius, as it must be.

PROBLEM 1.7 Filling a tank

Pipe A alone fills a tank 5 hours faster than pipe B alone. Together they fill it in 6 hours. How long does each take alone?

Solution. Let A take x hours, so B takes $x + 5$. In one hour A fills $\frac{1}{x}$ of the tank and B fills $\frac{1}{x+5}$; together they fill $\frac{1}{6}$:

$$\frac{1}{x} + \frac{1}{x+5} = \frac{1}{6} \implies 6(2x+5) = x(x+5) \implies x^2 - 7x - 30 = 0.$$

So $(x - 10)(x + 3) = 0$. A time cannot be negative, so $x = 10$: pipe A takes 10 hours and pipe B takes 15. Check: $\frac{1}{10} + \frac{1}{15} = \frac{1}{6}$.

SKILL DRILLS

Short problems for speed and accuracy. Answers follow the list.

D1.1 Write $\{x : -1 < x \leq 4\}$ in interval notation.

D1.2 Write $(-\infty, 2] \cap [-3, \infty)$ as one interval.

D1.3 Evaluate $|3 - \pi|$.

D1.4 Solve $5x - 7 = 2x + 8$.

D1.5 Solve $\frac{x}{4} + \frac{x}{6} = 5$.

D1.6 Solve $x^2 = 49$.

D1.7 Solve $x^2 - 9x + 20 = 0$.

D1.8 Solve $2x^2 - x - 3 = 0$.

D1.9 Solve $x^2 + 6x + 4 = 0$.

D1.10 Find the discriminant of $3x^2 - 5x + 4$.

D1.11 Solve $|x + 4| = 7$.

D1.12 Solve $\sqrt{x - 3} = 4$.

D1.13 Solve $-2x + 5 > 11$.

D1.14 Solve $|x - 1| < 3$.

D1.15 Solve $|2x + 3| \geq 7$.

D1.16 Solve $(x - 2)(x + 5) \leq 0$.

D1.17 Solve $\frac{x}{x-4} > 0$.

- D1.18** Find the distance from $(1, -2)$ to $(7, 6)$.
- D1.19** Find the midpoint of $(-3, 8)$ and $(5, -2)$.
- D1.20** Find the slope of the line through $(2, 5)$ and $(6, -3)$.
- D1.21** Find the line through $(1, 4)$ with slope -2 , in slope-intercept form.
- D1.22** Find the slope of a line perpendicular to $2x - 5y = 1$.
- D1.23** Find the center and radius of $(x + 1)^2 + (y - 6)^2 = 20$.
- D1.24** Write the circle with center $(0, -3)$ and radius 4.

ANSWERS TO SKILL DRILLS

D1.1 $(-1, 4]$. D1.2 $[-3, 2]$. D1.3 $\pi - 3$. D1.4 5. D1.5 12. D1.6 $-7, 7$. D1.7 4, 5. D1.8 $-1, \frac{3}{2}$. D1.9 $-3 - \sqrt{5}, -3 + \sqrt{5}$. D1.10 -23 . D1.11 $-11, 3$. D1.12 19. D1.13 $(-\infty, -3)$. D1.14 $(-2, 4)$. D1.15 $(-\infty, -5] \cup [2, \infty)$. D1.16 $[-5, 2]$. D1.17 $(-\infty, 0) \cup (4, \infty)$. D1.18 10. D1.19 $(1, 3)$. D1.20 -2 . D1.21 $y = -2x + 6$. D1.22 $-\frac{5}{2}$. D1.23 Center $(-1, 6)$, radius $2\sqrt{5}$. D1.24 $x^2 + (y + 3)^2 = 16$.

EXERCISES

COMPUTATION

- 1.1** Write as intervals: $\{x : -2 \leq x < 5\}$, $\{x : x > 3\} \cup \{x : x \leq -1\}$, $[0, 4] \cap (2, 7]$.
- 1.2** Evaluate $|\pi - 4|$, $|\sqrt{2} - 1|$, $|1 - \sqrt{2}|$, and $|x - 3|$ if $x < 3$.
- 1.3** Solve $\frac{x}{3} - \frac{x-2}{4} = 1$.
- 1.4** Solve $3x^2 - 7x + 2 = 0$ by factoring.
- 1.5** Solve $x^2 - 6x + 2 = 0$ by completing the square.
- 1.6** Solve $4x^2 + 4x + 1 = 0$ and $3x^2 - 2x + 4 = 0$, and state the discriminant of each.
- 1.7** Find k so that $x^2 + kx + 9 = 0$ has exactly one real solution.
- 1.8** Solve $\frac{1}{x} + \frac{2}{x+1} = 1$.
- 1.9** Solve $\frac{x}{x-3} = \frac{3}{x-3} + 2$.

1.10 Solve $x^4 - 13x^2 + 36 = 0$.

1.11 Solve $\sqrt{x+7} = x+1$.

1.12 Solve $\sqrt{2x+1} - \sqrt{x} = 1$.

1.13 Solve $|3x-1| = 8$ and $|x+2| = -1$.

1.14 Solve $|x^2 - 5| = 4$.

1.15 Solve $5 - 2x \geq 3x + 20$.

1.16 Solve $-3 < \frac{2x-1}{3} \leq 5$.

1.17 Solve $|2x-7| \leq 5$.

1.18 Solve $|4-x| > 2$.

1.19 Solve $x^2 - x - 12 < 0$.

1.20 Solve $2x^2 + 5x \geq 3$.

1.21 Solve $x^3 \geq 4x$.

1.22 Solve $\frac{x-4}{x+1} \geq 0$.

1.23 Solve $\frac{2x+1}{x-3} < 1$.

1.24 Solve $\frac{x^2-9}{x^2+1} \leq 0$.

1.25 Find the distance between $(-2, 5)$ and $(4, -3)$, and the midpoint.

1.26 Show that $(1, 1)$, $(4, 5)$, and $(8, 2)$ are vertices of an isosceles right triangle.

1.27 Find the center and radius of $x^2 + y^2 - 8x + 2y + 8 = 0$.

1.28 Find the line through $(2, -3)$ parallel to $3x - y = 5$, and the line through the same point perpendicular to it.

1.29 Find the equation of the circle with diameter from $(-1, 3)$ to $(5, -5)$.

1.30 Find the points where the line $y = x + 1$ meets the circle $x^2 + y^2 = 13$.

CONCEPTS AND PROOF

1.31 Prove that $\sqrt{3}$ is irrational. Where does the proof for $\sqrt{4}$ break down?

- 1.32** Prove that the sum of a rational and an irrational number is irrational. Is the sum of two irrationals always irrational?
- 1.33** Prove that $||a| - |b|| \leq |a - b|$.
- 1.34** Prove that if $a, b > 0$ then $\frac{a+b}{2} \geq \sqrt{ab}$ (the AM–GM inequality), with equality only when $a = b$.
- 1.35** Show that the solutions of $ax^2 + bx + c = 0$ have sum $-\frac{b}{a}$ and product $\frac{c}{a}$.
- 1.36** Explain why squaring both sides of an equation can introduce extraneous solutions but never loses a solution.
- 1.37** For which c does $x^2 + 4x + c > 0$ hold for every real x ?
- 1.38** Show that the set of points equidistant from $(a, 0)$ and $(-a, 0)$ is the y -axis, using the distance formula.
- 1.39** Prove that the diagonals of a parallelogram bisect each other, using coordinates.
- 1.40** Show that the distance from the point (x_0, y_0) to the line $Ax + By + C = 0$ is $\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$ for the special case of the line $y = mx$ and the point $(0, b)$.

CHALLENGE

- 1.41** Solve $|x - 1| + |x + 2| = 5$.
- 1.42** Solve $|x - 1| + |x + 2| < 3$, and explain the answer geometrically.
- 1.43** Find all real x with $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$.
- 1.44** Prove that $0.\overline{9} = 1$ by two different methods.
- 1.45** Find all pairs of positive integers (a, b) with $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$.

CHAPTER 2

FUNCTIONS AND THEIR GRAPHS

LEARNING OBJECTIVES

- Define a function, find domains and ranges, and evaluate functions, including piecewise functions and difference quotients.
- Read graphs: the vertical line test, intervals of increase and decrease, extreme values, and symmetry.
- Compute average rates of change, and interpret them as slopes of secant lines.
- Graph functions by transforming basic graphs with shifts, stretches, and reflections.
- Combine functions algebraically and by composition, and decompose composite functions.
- Find inverse functions, test for one-to-one functions, and graph inverses by reflection.

The idea of a function, a rule that assigns to each input exactly one output, is the most important idea in this book. Calculus is the study of how functions change, and every topic in the chapters ahead (polynomials, exponentials, trigonometry) is the study of a particular family of functions. This chapter develops the general language.

2.1 FUNCTIONS

Definition 2.1 (Function). A *function* f from a set D to a set E is a rule that assigns to each element x of D exactly one element $f(x)$ of E . D is the *domain*; the set of all values $f(x)$ is the *range*. x is the *independent variable* and $y = f(x)$ the *dependent variable*.

When a function is given by a formula and no domain is stated, the domain is the set of all real numbers for which the formula produces a real number. In practice: exclude zeros of denominators, and require even roots to have nonnegative arguments and logarithms to have positive arguments.

Example 2.2 (Domains). Find the domains of (a) $f(x) = \frac{x+1}{x^2-x-6}$, (b) $g(x) = \sqrt{4-x^2}$, (c) $h(x) = \frac{\sqrt{x+3}}{x-2}$.

Solution. (a) $x^2 - x - 6 = (x-3)(x+2) \neq 0$: domain $\{x : x \neq 3, -2\}$. (b) $4 - x^2 \geq 0$: domain $[-2, 2]$. The range is $[0, 2]$, since g is the upper half of the circle $x^2 + y^2 = 4$. (c) $x + 3 \geq 0$ and $x \neq 2$: domain $[-3, 2) \cup (2, \infty)$.

Example 2.3 (The difference quotient). For $f(x) = x^2 - 3x$, simplify $\frac{f(x+h) - f(x)}{h}$ for $h \neq 0$.

Solution. $f(x+h) = x^2 + 2xh + h^2 - 3x - 3h$. Subtracting $f(x)$ leaves $2xh + h^2 - 3h$, and dividing by h gives $2x + h - 3$.

The difference quotient is the slope of the line through $(x, f(x))$ and $(x+h, f(x+h))$. In calculus, letting h shrink to 0 turns it into the derivative; here, $2x + h - 3$ becomes $2x - 3$.

COMMON ERROR ($f(x+h) \neq f(x) + h$). $f(x+h)$ means: replace every x in the formula by $(x+h)$. For $f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$, which is neither $x^2 + h$ nor $x^2 + h^2$.

2.1.1 PIECEWISE FUNCTIONS

A function may be defined by different formulas on different parts of its domain. The absolute value is the basic example.

Example 2.4 (A piecewise cost function). A parking garage charges \$3 for the first hour or less and \$2 for each additional hour or part of an hour, up to a daily maximum of \$15. The cost $C(t)$ for t hours is a *step function*: $C(t) = 3$ for $0 < t \leq 1$, $C(t) = 5$ for $1 < t \leq 2$, and so on, reaching 15 for $t > 6$. Its graph is a staircase, and it jumps at every whole hour.

2.2 GRAPHS OF FUNCTIONS

The graph of f is the set of points $(x, f(x))$ for x in the domain. Because each input has exactly one output:

Theorem 2.5 (Vertical Line Test). *A curve in the plane is the graph of a function of x if and only if no vertical line meets it more than once.*

Definition 2.6 (Increasing, decreasing, extrema). f is *increasing* on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I , and *decreasing* if $f(x_1) > f(x_2)$. $f(c)$ is a *local maximum* if $f(c) \geq f(x)$ for all x near c , and the *absolute maximum* if $f(c) \geq f(x)$ for all x in the domain; minimum values are defined similarly.

Definition 2.7 (Symmetry). f is *even* if $f(-x) = f(x)$ for all x in its domain; its graph is symmetric about the y -axis. f is *odd* if $f(-x) = -f(x)$; its graph is symmetric about the origin.

Example 2.8 (Even, odd, or neither). $f(x) = x^4 - 3x^2$ is even. $g(x) = x^3 + \frac{1}{x}$ is odd. $h(x) = x^2 + x$ is neither: $h(-1) = 0$ while $h(1) = 2$ and $-h(1) = -2$.

Definition 2.9 (Average rate of change). The average rate of change of f between $x = a$ and $x = b$ is $\frac{f(b) - f(a)}{b - a}$, the slope of the secant line through $(a, f(a))$ and $(b, f(b))$.

Example 2.10 (Average velocity). A ball dropped from a tower has fallen $s(t) = 16t^2$ feet after t seconds. Its average velocity over $[1, 3]$ is $\frac{144 - 16}{2} = 64$ ft/s, and over $[2, 2.1]$ it is $\frac{16(4.41) - 64}{0.1} = 65.6$ ft/s. The averages over shorter and shorter intervals around $t = 2$ approach 64 ft/s, which calculus will call the instantaneous velocity.

2.3 TRANSFORMATIONS

A small library of basic graphs (x , x^2 , x^3 , $\sqrt{x}$, $|x|$, $\frac{1}{x}$, $\sqrt[3]{x}$) together with the transformations in the table below produces the graphs of a great many functions without plotting points.

Theorem 2.11 (Transformations of graphs). Let $c > 0$. Starting from $y = f(x)$:

Equation	Effect on the graph
$y = f(x) + c$, $y = f(x) - c$	shift up, down by c
$y = f(x - c)$, $y = f(x + c)$	shift right, left by c
$y = cf(x)$	stretch vertically by c (compress if $c < 1$)
$y = f(cx)$	compress horizontally by c (stretch if $c < 1$)
$y = -f(x)$	reflect in the x -axis
$y = f(-x)$	reflect in the y -axis

Changes *outside* f act on y the way you expect. Changes *inside* act on x in the opposite way: $f(x - 3)$ moves the graph right, because the output that f produced at x is now produced at $x + 3$.

Example 2.12 (Several transformations). Sketch $y = -2\sqrt{x+3} + 1$.

Solution. Start with $y = \sqrt{x}$. Shift left 3 ($\sqrt{x+3}$), stretch vertically by 2, reflect in the x -axis, then shift up 1. The graph starts at $(-3, 1)$ and falls to the right, passing through $(-2, -1)$ and $(1, -3)$.

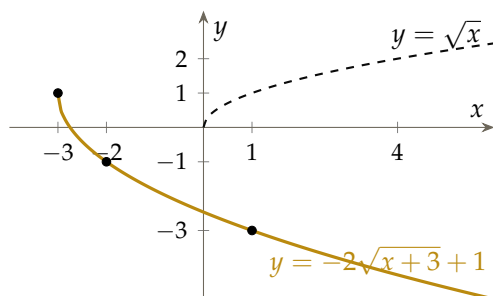


Figure 2.1: $y = -2\sqrt{x+3} + 1$ obtained from $y = \sqrt{x}$ by four transformations.

COMMON ERROR (Order of horizontal operations). $y = f(2x - 6) = f(2(x - 3))$ is f compressed by 2 and then shifted right 3, not right 6. Factor the coefficient out of the inside before reading the shift.

2.4 COMBINING FUNCTIONS

Functions can be added, subtracted, multiplied, and divided pointwise: $(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, and $(f/g)(x) = f(x)/g(x)$, on the intersection of their domains (and, for the quotient, where $g(x) \neq 0$).

Definition 2.13 (Composition). The *composition* of f and g is $(f \circ g)(x) = f(g(x))$. Its domain is the set of x in the domain of g for which $g(x)$ is in the domain of f .

Example 2.14 (Composition is not commutative). Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$. Find $f \circ g$, $g \circ f$, and their domains.

Solution. $(f \circ g)(x) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$, with domain $(-\infty, 2]$. $(g \circ f)(x) = \sqrt{2 - \sqrt{x}}$, which requires $x \geq 0$ and $\sqrt{x} \leq 2$: domain $[0, 4]$.

Example 2.15 (Decomposition). Write $F(x) = \cos^2(x + 9)$ as a composition of three simpler functions.

Solution. $F = f \circ g \circ h$ with $h(x) = x + 9$, $g(x) = \cos x$, $f(x) = x^2$. Recognizing such layers is exactly the skill needed for the chain rule in calculus.

2.5 INVERSE FUNCTIONS

Definition 2.16 (One-to-one and inverse). f is *one-to-one* if it never takes the same value twice: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. If f is one-to-one with domain A and range B , its *inverse* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

Theorem 2.17 (Properties of inverses). (1) Horizontal Line Test: f is one-to-one if and only if no horizontal line meets its graph more than once. (2)

$f^{-1}(f(x)) = x$ for x in A , and $f(f^{-1}(y)) = y$ for y in B . (3) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$. (4) A function that is increasing (or decreasing) on its whole domain is one-to-one.

Proof of (3). (a, b) is on the graph of $f \iff f(a) = b \iff f^{-1}(b) = a \iff (b, a)$ is on the graph of f^{-1} . And the reflection of (a, b) in $y = x$ is (b, a) : the segment joining them has slope -1 and midpoint $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ on the line. ■

Example 2.18 (Finding an inverse). Find the inverse of $f(x) = \frac{2x+1}{x-3}$.

Solution. Write $y = \frac{2x+1}{x-3}$ and solve for x : $y(x-3) = 2x+1$, so $xy - 2x = 3y + 1$ and $x = \frac{3y+1}{y-2}$. Interchanging the letters, $f^{-1}(x) = \frac{3x+1}{x-2}$, with domain $x \neq 2$ (which is the range of f). Check: $f^{-1}(f(0)) = f^{-1}\left(-\frac{1}{3}\right) = \frac{0}{-7/3} = 0$.

Example 2.19 (Restricting the domain). $f(x) = x^2$ is not one-to-one on $\mathbb{R}$, but on $[0, \infty)$ it is, with inverse $\sqrt{x}$. Restricting the domain in this way is how the inverse trigonometric functions are defined in Chapter 6.

COMMON ERROR ($f^{-1}(x) \neq \frac{1}{f(x)}$). The -1 in f^{-1} is not an exponent. For $f(x) = 2x$, $f^{-1}(x) = \frac{x}{2}$ while $\frac{1}{f(x)} = \frac{1}{2x}$.

LOOKING AHEAD. In calculus, the derivative of an inverse function is found from the reflection property: reflecting a graph in $y = x$ exchanges rise and run, so slopes at corresponding points are reciprocals, $(f^{-1})'(b) = \frac{1}{f'(a)}$. Composition is the structure behind the chain rule, $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$, the most used rule in all of differentiation.

2.6 WORKED PROBLEMS

These problems combine domains, composition, transformations, and inverses. Each solution includes a check.

PROBLEM 2.1 Domains of compositions

Let $f(x) = \sqrt{x-1}$ and $g(x) = \frac{1}{x-3}$. Find the domains of $f \circ g$ and $g \circ f$.

Solution. $(f \circ g)(x) = \sqrt{\frac{1}{x-3} - 1}$ needs $x \neq 3$ and $\frac{1}{x-3} \geq 1$. That forces $x - 3 > 0$, and then $1 \geq x - 3$. So $3 < x \leq 4$: the domain is $(3, 4]$.

$(g \circ f)(x) = \frac{1}{\sqrt{x-1} - 3}$ needs $x \geq 1$ and $\sqrt{x-1} \neq 3$, that is $x \neq 10$. The domain is $[1, 10) \cup (10, \infty)$. The two compositions have completely different domains, a reminder that $f \circ g$ and $g \circ f$ are different functions.

PROBLEM 2.2 A difference quotient that simplifies

Simplify $\frac{f(x+h) - f(x)}{h}$ for $f(x) = x^3$, and find what it approaches as $h \rightarrow 0$.

Solution. $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. Subtracting x^3 and dividing by h gives $3x^2 + 3xh + h^2$ for $h \neq 0$. As h shrinks to 0 this approaches $3x^2$, which is the slope of the tangent line to $y = x^3$, the derivative of Volume III.

PROBLEM 2.3 Joining two pieces

Find c so that the graph of $f(x) = \begin{cases} cx + 1, & x < 2 \\ x^2 - c, & x \geq 2 \end{cases}$ has no break at $x = 2$.

Solution. The left piece approaches $2c + 1$ as x approaches 2, and the right piece has value $4 - c$ there. They match when $2c + 1 = 4 - c$, so $c = 1$. Check: both pieces give 3 at $x = 2$.

PROBLEM 2.4 An inverse and its domain

Find the inverse of $f(x) = \frac{x+1}{2x-3}$, and the domain and range of each.

Solution. Solve $y = \frac{x+1}{2x-3}$ for x : $2xy - 3y = x + 1$, so $x(2y - 1) = 3y + 1$ and $x = \frac{3y+1}{2y-1}$. Hence $f^{-1}(x) = \frac{3x+1}{2x-1}$. The domain of f is $x \neq \frac{3}{2}$; its range is the domain of f^{-1} , $x \neq \frac{1}{2}$. (The value $\frac{1}{2}$ is the horizontal asymptote of f , which f never reaches.) Check: $f(2) = 3$ and $f^{-1}(3) = \frac{10}{5} = 2$.

PROBLEM 2.5 Building a formula from transformations

Starting from $y = \sqrt{x}$, reflect in the y -axis, then shift right 2 and up 3. Find the new formula and its domain.

Solution. Reflecting gives $y = \sqrt{-x}$. Shifting right replaces x by $x - 2$: $y = \sqrt{-(x - 2)} = \sqrt{2 - x}$. Shifting up adds 3: $y = \sqrt{2 - x} + 3$. The domain is $(-\infty, 2]$, the reflection of $[0, \infty)$ shifted right by 2. The endpoint $(0, 0)$ has moved to $(2, 3)$.

PROBLEM 2.6 Even and odd parts

Write $f(x) = x^3 + x^2 - 4x + 1$ as the sum of an even and an odd function.

Solution. The even part is $\frac{f(x)+f(-x)}{2}$ and the odd part is $\frac{f(x)-f(-x)}{2}$. Since $f(-x) = -x^3 + x^2 + 4x + 1$, the even part is $x^2 + 1$ and the odd part is $x^3 - 4x$. For a polynomial, these are simply the even-degree and odd-degree terms.

PROBLEM 2.7 Average velocity

A ball has height $h(t) = -16t^2 + 64t$ feet. Find its average velocity on $[1, 1 + h]$, and what this approaches as h shrinks.

Solution. $h(1) = 48$ and $h(1 + h) = -16(1 + 2h + h^2) + 64 + 64h = 48 + 32h - 16h^2$. The average velocity is $\frac{32h - 16h^2}{h} = 32 - 16h$, which approaches 32 ft/s. The ball is rising at 32 ft/s at $t = 1$.

PROBLEM 2.8 Solving for an unknown function

Let $f(x) = 2x + 3$. Find g with $f(g(x)) = 6x - 1$, and find k with $k(f(x)) = 4x + 5$.

Solution. $f(g(x)) = 2g(x) + 3 = 6x - 1$ gives $g(x) = 3x - 2$. For k , let $u = f(x) = 2x + 3$, so $x = \frac{u-3}{2}$. Then $k(u) = 4 \cdot \frac{u-3}{2} + 5 = 2u - 1$, so $k(x) = 2x - 1$. Check: $k(f(x)) = 2(2x + 3) - 1 = 4x + 5$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- D2.1** $f(x) = x^2 - 2x$. Find $f(-3)$.
- D2.2** $f(x) = \frac{x}{x+1}$. Find $f(2)$ and $f(a-1)$.
- D2.3** Domain of $\frac{1}{x^2-9}$.
- D2.4** Domain of $\sqrt{6-2x}$.
- D2.5** Domain of $\frac{\sqrt{x}}{x-4}$.
- D2.6** Is $x^4 - x^2$ even, odd, or neither?
- D2.7** Is $x^3 - \frac{1}{x}$ even, odd, or neither?
- D2.8** Average rate of change of x^2 on $[1, 3]$.
- D2.9** $f(x) = 3x - 1$, $g(x) = x^2$. Find $f(g(2))$.
- D2.10** With the same f, g , find $g(f(2))$.
- D2.11** With the same f, g , find $(f \circ g)(x)$.
- D2.12** With the same f, g , find $(g \circ f)(x)$.
- D2.13** Inverse of $f(x) = 5x + 2$.
- D2.14** Inverse of $f(x) = \frac{1}{x} + 1$.
- D2.15** Inverse of $f(x) = x^3 - 4$.
- D2.16** Shift of $y = x^2$ giving $y = (x - 5)^2$.
- D2.17** Describe $y = -|x| + 2$ relative to $y = |x|$.
- D2.18** Vertex of $y = (x + 1)^2 - 6$.
- D2.19** Range of $y = \sqrt{x} - 3$.
- D2.20** Range of $y = -(x - 2)^2 + 7$.
- D2.21** $f(x) = \begin{cases} x + 1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$. Find $f(-2) + f(2)$.
- D2.22** Simplify $\frac{f(x+h)-f(x)}{h}$ for $f(x) = 4x - 7$.
- D2.23** If $(2, -1)$ is on the graph of f , name a point on $y = f(x + 3)$.
- D2.24** If $(2, -1)$ is on the graph of f , name a point on the graph of f^{-1} .

ANSWERS TO SKILL DRILLS

D2.1 15. D2.2 $\frac{2}{3}; \frac{a-1}{a}$. D2.3 $x \neq \pm 3$. D2.4 $(-\infty, 3]$. D2.5 $[0, 4) \cup (4, \infty)$. D2.6 Even. D2.7 Odd. D2.8 4. D2.9 11. D2.10 25. D2.11 $3x^2 - 1$. D2.12 $(3x - 1)^2$. D2.13 $\frac{x-2}{5}$. D2.14 $\frac{1}{x-1}$. D2.15 $\sqrt[3]{x} + 4$. D2.16 Right 5. D2.17 Reflected in the x -axis, up 2. D2.18 $(-1, -6)$. D2.19 $[-3, \infty)$. D2.20 $(-\infty, 7]$. D2.21 3. D2.22 4. D2.23 $(-1, -1)$. D2.24 $(-1, 2)$.

EXERCISES

COMPUTATION

2.1 If $f(x) = 2x^2 - 3x + 1$, find $f(-2)$, $f(a + 1)$, $f(2a)$, and $2f(a)$.

2.2 Find the domain of $f(x) = \frac{1}{\sqrt{x^2 - 5x}}$.

- 2.3** Find the domain of $g(t) = \sqrt{3-t} - \sqrt{2+t}$.
- 2.4** Find the domain and range of $h(x) = \sqrt{9-x^2} + 1$.
- 2.5** Simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = 3x^2 - x + 5$.
- 2.6** Simplify the difference quotient for $f(x) = \frac{1}{x+1}$.
- 2.7** Simplify the difference quotient for $f(x) = \sqrt{x}$ by rationalizing the numerator.
- 2.8** Evaluate $f(-3)$, $f(0)$, $f(2)$ for $f(x) = \begin{cases} x^2, & x < 0 \\ 2x+1, & 0 \leq x \leq 2, \\ 7, & x > 2 \end{cases}$ and sketch the graph.
- 2.9** Determine whether each function is even, odd, or neither: $x^5 + x$; $1 - x^4$; $2x - x^2$; $\frac{x}{x^2+1}$; $|x| + x$.
- 2.10** Find the average rate of change of $f(x) = x^3 - 4x$ on $[0, 2]$ and on $[1, 1+h]$.
- 2.11** Find the average rate of change of $f(x) = \frac{1}{x}$ between $x = a$ and $x = b$.
- 2.12** Describe how to obtain each graph from $y = x^2$: $y = (x+2)^2 - 3$; $y = -\frac{1}{2}x^2 + 4$; $y = (2x-4)^2$.
- 2.13** Sketch $y = |x-2| - 1$ and $y = 1 - |x+1|$.
- 2.14** Sketch $y = \frac{1}{x-3} + 2$ and identify its asymptotes.
- 2.15** The point $(2, 5)$ is on the graph of f . Find a point on the graph of $y = 3f(x-1) + 2$ and on $y = f(-2x)$.
- 2.16** Find $f+g$, fg , f/g and their domains for $f(x) = \sqrt{x+2}$, $g(x) = x^2 - 1$.
- 2.17** Find $f \circ g$, $g \circ f$, $f \circ f$ for $f(x) = 2x - 3$, $g(x) = x^2 + 1$.
- 2.18** Find $f \circ g$ and its domain for $f(x) = \frac{1}{x-1}$, $g(x) = \frac{2}{x}$.
- 2.19** Express $F(x) = \sqrt{1+\sqrt{x}}$ and $G(x) = \frac{1}{(x^2+1)^3}$ as compositions $f \circ g$.
- 2.20** Use the horizontal line test to decide whether $x^3 - x$, $x^3 + x$, and $|x|$ are one-to-one.

2.21 Find the inverse of $f(x) = 4x - 7$ and verify $f(f^{-1}(x)) = x$.

2.22 Find the inverse of $f(x) = \sqrt[3]{2x + 1}$.

2.23 Find the inverse of $f(x) = \frac{x}{x + 2}$, and state its domain and range.

2.24 Find the inverse of $f(x) = x^2 - 4x + 1$ restricted to $x \geq 2$.

2.25 If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

CONCEPTS AND PROOF

2.26 Prove that the sum of two odd functions is odd, the product of two odd functions is even, and the product of an odd and an even function is odd.

2.27 Prove that every function f defined on all of $\mathbb{R}$ can be written as the sum of an even and an odd function. *Hint:* $\frac{f(x) + f(-x)}{2}$.

2.28 Prove that a function that is increasing on its domain is one-to-one, and that its inverse is increasing.

2.29 Explain why an even function (other than one with domain $\{0\}$) cannot be one-to-one.

2.30 Show that $f(x) = \frac{ax + b}{cx - a}$ is its own inverse whenever $a^2 + bc \neq 0$.

2.31 Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ when f and g are one-to-one.

2.32 Is composition associative? Prove that $(f \circ g) \circ h = f \circ (g \circ h)$.

2.33 Explain, with a graph, why $y = f(x - c)$ moves the graph to the right when $c > 0$.

2.34 A function satisfies $f(x + y) = f(x) + f(y)$ for all x, y . Show that $f(0) = 0$ and that f is odd.

2.35 The function $T(x)$ gives the tax on income x : 10% on the first \$10,000, 20% on income above that. Write T as a piecewise function and find its inverse on $[0, \infty)$.

CHALLENGE

2.36 Find all functions of the form $f(x) = ax + b$ with $f(f(x)) = 4x + 9$.

2.37 Let $f(x) = \frac{1}{1-x}$. Compute $f(f(x))$ and $f(f(f(x)))$, and find $f^{(2026)}(x)$, the 2026-fold composition.

2.38 Find the domain of $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$.

2.39 Find all functions f defined for $x \neq 0, 1$ that satisfy $f(x) + f\left(\frac{x-1}{x}\right) = 1 + x$.

2.40 Sketch the graph of $y = \lfloor x \rfloor + \lfloor -x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

CHAPTER 3

POLYNOMIAL AND RATIONAL FUNCTIONS

LEARNING OBJECTIVES

- Write quadratic functions in vertex form and solve optimization problems with them.
- Describe the end behavior, zeros, and multiplicities of polynomial functions, and sketch their graphs.
- Divide polynomials by long and synthetic division, and apply the Remainder and Factor Theorems.
- Find rational zeros, and use the Fundamental Theorem of Algebra to factor polynomials completely over the complex numbers.
- Find the asymptotes and intercepts of rational functions and sketch their graphs.

Polynomials are the functions built from the variable and constants using only addition, subtraction, and multiplication. They are the simplest functions there are, and in calculus they are used to approximate everything else. Rational functions, quotients of polynomials, add one new behavior: vertical asymptotes, where the function blows up.

3.1 QUADRATIC FUNCTIONS

Completing the square puts every quadratic $f(x) = ax^2 + bx + c$ ($a \neq 0$) in *vertex form*

$$f(x) = a(x - h)^2 + k, \quad h = -\frac{b}{2a}, \quad k = f(h).$$

The graph is a parabola with vertex (h, k) and axis $x = h$, opening up if $a > 0$ and down if $a < 0$. So k is the minimum value of f when $a > 0$ and the maximum

when $a < 0$.

Example 3.1 (Vertex form). $f(x) = 2x^2 - 12x + 23 = 2(x^2 - 6x + 9) + 23 - 18 = 2(x - 3)^2 + 5$. Vertex $(3, 5)$; minimum value 5; range $[5, \infty)$.

Example 3.2 (Maximizing area). A rancher has 1200 m of fence to enclose a rectangular field and divide it into two pens with a fence parallel to one side. What dimensions maximize the total area?

Solution. Let x be the length of each of the three parallel sides and y the other side. Then $3x + 2y = 1200$, so $y = 600 - \frac{3}{2}x$ and the area is $A(x) = x(600 - \frac{3}{2}x) = -\frac{3}{2}x^2 + 600x$. The vertex is at $x = -\frac{600}{2(-3/2)} = 200$, giving $y = 300$ and $A = 60,000 \text{ m}^2$.

3.2 POLYNOMIAL FUNCTIONS AND THEIR GRAPHS

A polynomial of degree n is $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $a_n \neq 0$. a_n is the *leading coefficient*.

Theorem 3.3 (End behavior). For $|x|$ large, $P(x)$ behaves like its leading term a_nx^n . If n is even, both ends go up when $a_n > 0$ and both go down when $a_n < 0$. If n is odd, the ends go in opposite directions: down on the left and up on the right when $a_n > 0$.

Proof. For $x \neq 0$, $P(x) = a_nx^n \left(1 + \frac{a_{n-1}}{a_nx} + \cdots + \frac{a_0}{a_nx^n}\right)$, and the parenthesis approaches 1 as $|x|$ grows. ■

If $(x - c)^m$ is a factor of P but $(x - c)^{m+1}$ is not, c is a zero of *multiplicity* m . The graph *crosses* the axis at a zero of odd multiplicity and *touches* without crossing at a zero of even multiplicity; the higher the multiplicity, the flatter the graph is near the zero. A polynomial of degree n has at most n real zeros and at most $n - 1$ turning points.

Example 3.4 (Sketching a polynomial). Sketch $P(x) = -x(x - 2)^2(x + 3)$.

Solution. Degree 4, leading coefficient -1 : both ends go down. Zeros: -3 and 0 (multiplicity 1, crossing) and 2 (multiplicity 2, touching). Signs: $P(-4) = -144 < 0$, $P(-1) = 18 > 0$, $P(1) = -4 < 0$, $P(3) = -18 < 0$. The graph rises from the left, crosses at -3 , peaks, comes down through

the origin, dips below the axis, touches it at 2, and falls away.

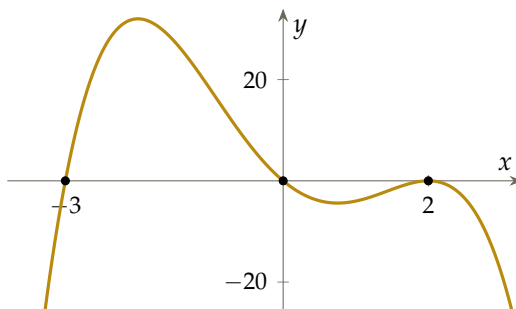


Figure 3.1: $P(x) = -x(x-2)^2(x+3)$ crosses the axis at -3 and 0 and touches it at 2 .

3.3 DIVIDING POLYNOMIALS

Theorem 3.5 (Division Algorithm). *If P and D are polynomials with $D \neq 0$, there are unique polynomials Q (the quotient) and R (the remainder) with*

$$P(x) = D(x)Q(x) + R(x),$$

where $R = 0$ or $\deg R < \deg D$.

Example 3.6 (Long division). Divide $P(x) = 6x^4 - x^3 + 4x^2 + x - 2$ by $D(x) = 2x^2 + 1$.

Solution. As in arithmetic, divide leading terms, multiply back, and subtract. $6x^4 \div 2x^2 = 3x^2$; subtracting $3x^2(2x^2 + 1) = 6x^4 + 3x^2$ leaves $-x^3 + x^2 + x - 2$. Next, $-x^3 \div 2x^2 = -\frac{1}{2}x$; subtracting $-\frac{1}{2}x(2x^2 + 1)$ leaves $x^2 + \frac{3}{2}x - 2$. Finally $x^2 \div 2x^2 = \frac{1}{2}$; subtracting $\frac{1}{2}(2x^2 + 1)$ leaves $\frac{3}{2}x - \frac{5}{2}$, of lower degree than D . So

$$6x^4 - x^3 + 4x^2 + x - 2 = (2x^2 + 1) \left(3x^2 - \frac{1}{2}x + \frac{1}{2} \right) + \frac{3}{2}x - \frac{5}{2}.$$

Theorem 3.7 (Remainder and Factor Theorems). *The remainder when $P(x)$ is divided by $x - c$ is $P(c)$. Consequently, $x - c$ is a factor of $P(x)$ if and only if $P(c) = 0$.*

Proof. Dividing by $x - c$ (degree 1) leaves a constant remainder r : $P(x) = (x -$

$c)Q(x) + r$. Setting $x = c$ gives $P(c) = r$. ■

Synthetic division is a compact algorithm for dividing by $x - c$: bring down the leading coefficient, multiply by c , add to the next coefficient, and repeat. The last number is the remainder $P(c)$.

Example 3.8 (Synthetic division). Divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Solution. Coefficients 2, -7, 0, 5 (the missing x term gets a 0):

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -4 \end{array}$$

So $2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) - 4$, and $P(3) = -4$.

3.4 ZEROS OF POLYNOMIALS

Theorem 3.9 (Rational Zeros Theorem). If $P(x) = a_nx^n + \cdots + a_0$ has integer coefficients and $\frac{p}{q}$ (in lowest terms) is a zero, then p divides a_0 and q divides a_n .

Proof. Substitute and multiply by q^n : $a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n = 0$. Every term except the last contains p , so p divides $a_0 q^n$; since p and q share no factor, p divides a_0 . Similarly every term except the first contains q , so q divides a_n . ■

Example 3.10 (Factoring completely). Factor $P(x) = 2x^3 + x^2 - 13x + 6$.

Solution. Candidates: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$. Testing, $P(2) = 16 + 4 - 26 + 6 = 0$. Synthetic division by $x - 2$ gives $2x^2 + 5x - 3 = (2x - 1)(x + 3)$. So $P(x) = (x - 2)(2x - 1)(x + 3)$, with zeros $2, \frac{1}{2}, -3$.

3.4.1 COMPLEX ZEROS

The polynomial $x^2 + 1$ has no real zeros. Introducing the number i with $i^2 = -1$, it factors as $(x - i)(x + i)$. A *complex number* has the form $a + bi$ with a, b real; complex numbers are added and multiplied like polynomials in i , using $i^2 = -1$. They are developed fully in Chapter 9. In 1799 Gauss proved that complex numbers are enough to factor every polynomial.

Theorem 3.11 (Fundamental Theorem of Algebra). *Every polynomial of degree $n \geq 1$ with complex coefficients has at least one complex zero. Consequently it factors completely as*

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n),$$

and has exactly n zeros counted with multiplicity.

The first statement is deep; its proofs use analysis or topology. The second follows by repeatedly applying the first and the Factor Theorem.

Theorem 3.12 (Conjugate Zeros). *If a polynomial has real coefficients and $a + bi$ is a zero, then so is its conjugate $a - bi$. Hence every real polynomial factors into real linear factors and irreducible real quadratic factors.*

Example 3.13 (Using a complex zero). Given that $2 + i$ is a zero of $P(x) = x^3 - 5x^2 + 9x - 5$, factor P completely.

Solution. By Theorem 3.12, $2 - i$ is also a zero, so $(x - (2 + i))(x - (2 - i)) = (x - 2)^2 - i^2 = x^2 - 4x + 5$ is a factor. Dividing, $P(x) = (x^2 - 4x + 5)(x - 1)$. Over the complex numbers, $P(x) = (x - 1)(x - 2 - i)(x - 2 + i)$.

3.5 RATIONAL FUNCTIONS

A rational function is $r(x) = \frac{P(x)}{Q(x)}$ with P, Q polynomials. Assume P and Q have no common factor (otherwise cancel it, noting a hole in the graph).

Theorem 3.14 (Asymptotes of rational functions). Let $r(x) = \frac{a_n x^n + \cdots}{b_m x^m + \cdots}$ in lowest terms.

- Vertical asymptotes are the lines $x = c$ where $Q(c) = 0$.
- If $n < m$, the horizontal asymptote is $y = 0$; if $n = m$, it is $y = \frac{a_n}{b_m}$; if $n > m$, there is none.
- If $n = m + 1$, there is a slant asymptote $y = mx + b$, the quotient when P is divided by Q .

Example 3.15 (A complete sketch). Sketch $r(x) = \frac{2x^2 - 8}{x^2 - x - 6}$.

Solution. Factor: $r(x) = \frac{2(x-2)(x+2)}{(x-3)(x+2)} = \frac{2(x-2)}{x-3}$ for $x \neq -2$. There is a hole at $x = -2$ (where the simplified function equals $\frac{2(-4)}{-5} = \frac{8}{5}$), a vertical asymptote $x = 3$, horizontal asymptote $y = 2$, x -intercept 2, and y -intercept $\frac{4}{3}$. As $x \rightarrow 3^+$ the function is large and positive; as $x \rightarrow 3^-$, large and negative.

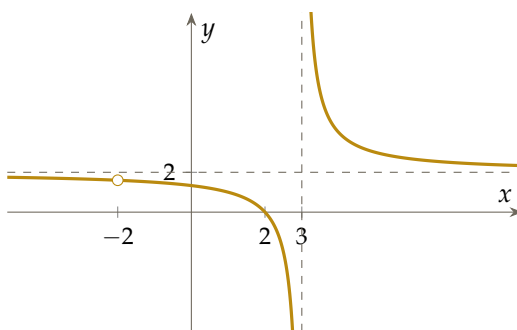


Figure 3.2: $r(x) = \frac{2x^2-8}{x^2-x-6}$: a hole at $x = -2$, asymptotes $x = 3$ and $y = 2$.

COMMON ERROR (Graphs can cross horizontal asymptotes). A graph never crosses a vertical asymptote, but it may cross a horizontal one, often several times. The horizontal asymptote describes only the behavior as $|x| \rightarrow \infty$. For example, $\frac{x}{x^2+1}$ crosses its asymptote $y = 0$ at the origin.

LOOKING AHEAD. The asymptotes found here by algebra are limits in disguise: a vertical asymptote at $x = c$ means $\lim_{x \rightarrow c} |r(x)| = \infty$, and a horizontal asymptote $y = L$ means $\lim_{x \rightarrow \infty} r(x) = L$. In calculus, partial fractions, which decompose a rational function into simple pieces using exactly the factorization theory of this chapter, are the method for integrating every rational function.

3.6 WORKED PROBLEMS

These problems combine the division algorithm, the zero theorems, and the analysis of graphs.

PROBLEM 3.1 A polynomial from its zeros

Find the cubic with zeros -1 (multiplicity 2) and 2 , whose graph passes through $(0, 4)$.

Solution. $P(x) = a(x+1)^2(x-2)$. At $x = 0$: $a(1)(-2) = 4$, so $a = -2$ and $P(x) = -2(x+1)^2(x-2) = -2x^3 + 6x + 4$. The graph touches the axis at -1 , crosses at 2 , rises on the left and falls on the right (degree 3, negative leading coefficient).

PROBLEM 3.2 Factoring a quartic completely

Factor $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$.

Solution. The possible rational zeros are the divisors of 12. $P(-1) = 1 + 2 - 7 - 8 + 12 = 0$, so $x + 1$ is a factor; synthetic division leaves $x^3 - 3x^2 - 4x + 12$. Group: $x^2(x-3) - 4(x-3) = (x-3)(x^2-4)$. So

$$P(x) = (x+1)(x-3)(x-2)(x+2).$$

PROBLEM 3.3 Two conditions, two unknowns

$P(x) = x^3 + ax^2 + bx + 6$ has $x - 1$ as a factor and leaves remainder 12 when divided by $x - 2$. Find a , b , and the other zeros.

Solution. By the Factor and Remainder Theorems: $P(1) = 7 + a + b = 0$ and $P(2) = 14 + 4a + 2b = 12$. So $a + b = -7$ and $2a + b = -1$, giving $a = 6$, $b = -13$. Then $P(x) = x^3 + 6x^2 - 13x + 6 = (x-1)(x^2 + 7x - 6)$, and the other zeros are $\frac{-7 \pm \sqrt{73}}{2}$.

PROBLEM 3.4 Using a complex zero

Given that $2i$ is a zero of $P(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$, find all zeros.

Solution. Real coefficients force $-2i$ to be a zero too, so $(x-2i)(x+2i) = x^2 + 4$ divides P . Long division gives $P(x) = (x^2 + 4)(x^2 - 4x + 5)$. The second factor has zeros $\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$. The four zeros are $\pm 2i$ and $2 \pm i$; P has no real zeros, so its graph never meets the x -axis.

PROBLEM 3.5 A complete rational graph

Analyze $r(x) = \frac{x^2 - 1}{x^2 - 4}$.

Solution. r is even. Intercepts: $x = \pm 1$ and $y = \frac{1}{4}$. Vertical asymptotes $x = \pm 2$; horizontal asymptote $y = 1$ (equal degrees, leading coefficients 1). Sign of $\frac{(x-1)(x+1)}{(x-2)(x+2)}$: positive on $(-\infty, -2)$, negative on $(-2, -1)$, positive on $(-1, 1)$, negative on $(1, 2)$, positive on $(2, \infty)$. Near $x = 2$: from the right $r \rightarrow +\infty$, from the left $r \rightarrow -\infty$. The graph has a hump through $(0, \frac{1}{4})$ between the zeros, and approaches $y = 1$ from above at both ends, since $r(x) - 1 = \frac{3}{x^2 - 4} > 0$ for $|x| > 2$.

PROBLEM 3.6 Maximizing revenue

A theater sells 300 tickets at \$10. Each \$0.50 increase in price loses 10 customers. What price maximizes revenue?

Solution. After n increases, the price is $10 + 0.5n$ and attendance $300 - 10n$, so revenue is $R(n) = (10 + 0.5n)(300 - 10n) = 3000 + 50n - 5n^2$. The vertex is at $n = \frac{-50}{2(-5)} = 5$: price \$12.50, attendance 250, revenue \$3125.

PROBLEM 3.7 A cubic inequality

Solve $x^3 - 3x^2 - 4x + 12 \leq 0$.

Solution. Factor by grouping: $x^2(x - 3) - 4(x - 3) = (x - 3)(x - 2)(x + 2)$. The sign is negative on $(-\infty, -2)$, positive on $(-2, 2)$, negative on $(2, 3)$, positive on $(3, \infty)$. Including the zeros: $(-\infty, -2] \cup [2, 3]$.

PROBLEM 3.8 Conjugate pairs and degree

Find a polynomial of least degree with real coefficients, leading coefficient 1, and zeros 3 and $1 - 2i$.

Solution. $1 + 2i$ must also be a zero. $(x - (1 - 2i))(x - (1 + 2i)) = (x - 1)^2 + 4 = x^2 - 2x + 5$. So $P(x) = (x - 3)(x^2 - 2x + 5) = x^3 - 5x^2 + 11x - 15$. Degree 2 is impossible: a real quadratic with a nonreal zero has both conjugates as zeros, leaving no room for 3.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|---|--|
| D3.1 Vertex of $y = x^2 - 6x + 5$. | D3.12 Possible rational zeros of $x^3 - 2x + 4$. |
| D3.2 Vertex form of $y = 2x^2 + 8x + 3$. | D3.13 Possible rational zeros of $3x^2 + x - 2$. |
| D3.3 Maximum value of $y = -x^2 + 4x + 1$. | D3.14 Factor $x^3 - 8$. |
| D3.4 Zeros of $x^2 - x - 12$. | D3.15 Factor $x^3 + x^2 - x - 1$. |
| D3.5 End behavior of $-3x^4 + x$. | D3.16 Zeros of $x^2 + 9$. |
| D3.6 End behavior of $2x^3 - 5$. | D3.17 Zeros of $x^2 - 2x + 2$. |
| D3.7 Zeros and multiplicities of $x^3(x - 2)^2$. | D3.18 Number of complex zeros of $x^7 - 1$, with multiplicity. |
| D3.8 Remainder when $x^3 - 2x + 1$ is divided by $x - 2$. | D3.19 Vertical asymptotes of $\frac{x}{x^2 - 9}$. |
| D3.9 Remainder when $x^4 + 1$ is divided by $x + 1$. | D3.20 Horizontal asymptote of $\frac{3x^2 + 1}{x^2 - 4}$. |
| D3.10 Quotient of $x^2 + 5x + 6$ divided by $x + 2$. | D3.21 Horizontal asymptote of $\frac{x + 1}{x^2 + 1}$. |
| D3.11 Is $x + 3$ a factor of $x^3 + 27$? | D3.22 Slant asymptote of $\frac{x^2 + 1}{x}$. |
| | D3.23 Hole in the graph of $\frac{x^2 - 1}{x - 1}$. |
| | D3.24 y -intercept of $\frac{2x - 6}{x + 3}$. |

ANSWERS TO SKILL DRILLS

D3.1 $(3, -4)$. D3.2 $2(x + 2)^2 - 5$. D3.3 5. D3.4 $4, -3$. D3.5 Both ends down.
D3.6 Down on the left, up on the right. D3.7 0 (mult. 3), 2 (mult. 2). D3.8 5.
D3.9 2. D3.10 $x + 3$. D3.11 Yes. D3.12 $\pm 1, \pm 2, \pm 4$. D3.13 $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$.
D3.14 $(x - 2)(x^2 + 2x + 4)$. D3.15 $(x - 1)(x + 1)^2$. D3.16 $\pm 3i$. D3.17 $1 \pm i$.
D3.18 7. D3.19 $x = \pm 3$. D3.20 $y = 3$. D3.21 $y = 0$. D3.22 $y = x$. D3.23
 $(1, 2)$. D3.24 -2 .

EXERCISES

COMPUTATION

- 3.1** Write $f(x) = -3x^2 + 12x - 7$ in vertex form, and give its vertex and range.
- 3.2** Find the quadratic with vertex $(2, -3)$ passing through $(4, 5)$.
- 3.3** A ball is thrown upward with height $h(t) = -16t^2 + 48t + 6$ feet. Find its maximum height and when it occurs.
- 3.4** Find two numbers whose sum is 30 and whose product is as large as possible.
- 3.5** A store sells 200 shirts a week at \$20 each; each \$1 price cut sells 20 more. What price maximizes revenue?
- 3.6** Describe the end behavior of $P(x) = -2x^5 + x^3 - 4$ and of $Q(x) = 3x^4 - 100x^3$.
- 3.7** Find the zeros and their multiplicities, and sketch: $P(x) = x^2(x - 3)(x + 1)^3$.
- 3.8** Find a polynomial of degree 3 with zeros $-1, 2, 4$ and $P(0) = 16$.
- 3.9** Divide $x^4 - 2x^3 + 3x - 5$ by $x^2 + 1$.
- 3.10** Use synthetic division to divide $x^4 - 3x^3 + 2x - 8$ by $x + 2$.
- 3.11** Use the Remainder Theorem to find $P(-3)$ for $P(x) = 2x^5 - 3x^4 + x^2 + 7$.
- 3.12** Find k so that $x - 2$ is a factor of $x^3 + kx^2 - 4x + 8$.
- 3.13** List the possible rational zeros of $2x^4 - 3x^3 + x - 6$.
- 3.14** Factor completely: $x^3 - 2x^2 - 5x + 6$.
- 3.15** Factor completely: $2x^3 - 9x^2 + 7x + 6$.
- 3.16** Find all zeros of $x^4 - x^3 - 7x^2 + x + 6$.
- 3.17** Find all real and complex zeros of $x^3 - 1$ and of $x^4 - 16$.
- 3.18** Given that $1 - 2i$ is a zero of $x^3 - x^2 + 3x + 5$, factor completely.
- 3.19** Find a polynomial with real coefficients of degree 4 with zeros i and $3 - i$ and leading coefficient 1.
- 3.20** Find the asymptotes and intercepts of $r(x) = \frac{3x - 1}{x + 2}$.
- 3.21** Find the asymptotes, intercepts, and any holes of $r(x) = \frac{x^2 - x - 2}{x^2 - 4}$.

3.22 Find all asymptotes of $r(x) = \frac{x^2 + 2x - 3}{x - 2}$.

3.23 Find all asymptotes of $r(x) = \frac{x^3 + 1}{x^2 + x}$.

3.24 Sketch $r(x) = \frac{x}{x^2 - 4}$, including the sign of r on each interval.

3.25 Find a rational function with vertical asymptotes $x = \pm 2$, horizontal asymptote $y = 3$, and x -intercept 1 (only).

CONCEPTS AND PROOF

3.26 Prove that the vertex of $y = ax^2 + bx + c$ is at $x = -\frac{b}{2a}$ by completing the square.

3.27 Prove that a polynomial of degree n has at most n distinct zeros.

3.28 Show that $x^3 - x - 1$ has no rational zeros but does have a real zero between 1 and 2.

3.29 Prove that $\sqrt[3]{2}$ is irrational using the Rational Zeros Theorem.

3.30 Prove the Conjugate Zeros Theorem: if P has real coefficients and $P(z) = 0$, then $P(\bar{z}) = 0$. Use $\overline{z\bar{w}} = \bar{z}w$ and $\overline{z + w} = \bar{z} + \bar{w}$.

3.31 Explain why every polynomial of odd degree with real coefficients has at least one real zero.

3.32 If $P(x)$ is divided by $(x - 1)(x - 2)$, the remainder is $ax + b$. If $P(1) = 3$ and $P(2) = 7$, find a and b .

3.33 Find the slant asymptote of $r(x) = \frac{x^3 + x^2}{x^2 + 1}$ and the point where the graph crosses it.

3.34 Explain why a polynomial of degree n has at most $n - 1$ turning points.

3.35 Give an example of a rational function whose graph has a hole and no vertical asymptote.

CHALLENGE

3.36 Find all polynomials P with $P(x + 1) - P(x) = 2x + 1$ for all x and $P(0) = 0$.

3.37 The polynomial $x^3 + ax^2 + bx + c$ has zeros r, s, t . Express $r^2 + s^2 + t^2$ in terms of a, b, c .

- 3.38** Find the remainder when x^{100} is divided by $x^2 - 1$.
- 3.39** Verify Sophie Germain's identity $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$, and use it to prove that $n^4 + 4$ is not prime for any integer $n > 1$.
- 3.40** A box with an open top is made from a 20×30 cm sheet by cutting squares of side x from the corners. For what x is the volume exactly 1000 cm^3 ? (Find all solutions in the physical domain.)

CHAPTER 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LEARNING OBJECTIVES

- Graph exponential functions and use them to model growth, decay, and compound interest.
- Define e through continuous compounding, and work with the natural exponential function.
- Define logarithms as inverses of exponentials, graph them, and evaluate them.
- Prove and apply the laws of logarithms and the change-of-base formula.
- Solve exponential and logarithmic equations, and check for extraneous solutions.
- Model with exponential growth and decay, half-life, and logarithmic scales.

Polynomials grow, but exponential functions grow faster than any polynomial. Populations, investments, epidemics, and chain reactions grow exponentially; radioactive material, drug concentrations, and the temperature difference between a cooling cup and its room decay exponentially. Logarithms, the inverses of exponentials, turn multiplication into addition and make these processes tractable.

4.1 EXPONENTIAL FUNCTIONS

Definition 4.1 (Exponential function). For a constant $a > 0$, $a \neq 1$, the *exponential function with base a* is $f(x) = a^x$.

For rational $x = p/q$, $a^x = \sqrt[q]{a^p}$. For irrational x , a^x is defined by approximating x with rationals: $2^{\sqrt{2}}$ is the number that $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, $\dots$ approach. (A rigorous definition is given in Volume III using integrals.) The laws of exponents $a^{x+y} = a^x a^y$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$ hold for all real exponents.

Theorem 4.2 (Graphs of exponential functions). $f(x) = a^x$ has domain $\mathbb{R}$, range $(0, \infty)$, and $f(0) = 1$. If $a > 1$, f is increasing with $\lim_{x \rightarrow -\infty} a^x = 0$; if $0 < a < 1$, f is decreasing with $\lim_{x \rightarrow \infty} a^x = 0$. In both cases the x -axis is a horizontal asymptote.

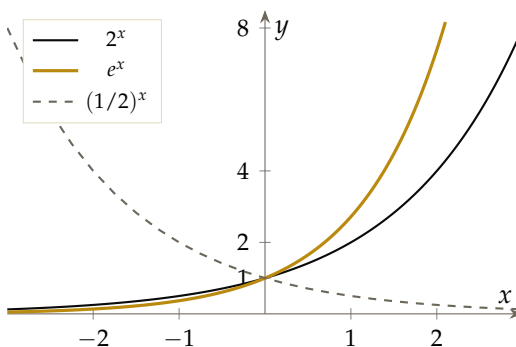


Figure 4.1: Exponential functions. All pass through $(0, 1)$; $(1/2)^x$ is the reflection of 2^x in the y -axis.

Example 4.3 (Exponential versus power). Compare $f(x) = 2^x$ and $g(x) = x^2$. They are equal at $x = 2$ and $x = 4$ (and at one negative x), with $x^2 > 2^x$ on $(2, 4)$. But for $x > 4$ the exponential wins and the gap explodes: $2^{10} = 1024$ while $10^2 = 100$, and $2^{20} = 1,048,576$ while $20^2 = 400$. Every exponential with base greater than 1 eventually overtakes every polynomial.

4.1.1 COMPOUND INTEREST AND THE NUMBER e

A principal P invested at annual rate r compounded n times per year grows to

$$A(t) = P \left(1 + \frac{r}{n} \right)^{nt}$$

after t years: each period multiplies the balance by $1 + \frac{r}{n}$, and there are nt periods. As compounding becomes more frequent, the growth factor for one year at 100% interest is $\left(1 + \frac{1}{n} \right)^n$:

n	1	4	12	365	10,000	1,000,000
$\left(1 + \frac{1}{n} \right)^n$	2	2.441406	2.613035	2.714567	2.718146	2.718280

The values increase toward a limit.

Definition 4.4 (The number e). $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = 2.718281828459 \dots$
The function e^x is the *natural exponential function*.

With continuous compounding, the balance is $A(t) = Pe^{rt}$. In calculus, e is singled out by a different property: the graph of e^x has slope exactly 1 at $x = 0$, and the slope of e^x at every point equals its value.

Example 4.5 (Comparing compounding). \$5000 at 6% for 10 years: compounded annually, $5000(1.06)^{10} = \$8954.24$; monthly, $5000(1.005)^{120} = \$9096.98$; continuously, $5000e^{0.6} = \$9110.59$.

4.2 LOGARITHMIC FUNCTIONS

Because a^x is increasing (or decreasing), it is one-to-one and has an inverse.

Definition 4.6 (Logarithm). For $a > 0$, $a \neq 1$, the *logarithm with base a* is the inverse of a^x :

$$\log_a x = y \iff a^y = x \quad (x > 0).$$

$\log_{10} x$ is written $\log x$ (the *common logarithm*) and $\log_e x$ is written $\ln x$ (the *natural logarithm*).

In words, $\log_a x$ is the *exponent you must put on a to get x* . So $\log_2 8 = 3$, $\log_{10} 0.001 = -3$, $\log_9 3 = \frac{1}{2}$, and $\ln e^5 = 5$. Because they are inverse functions,

$$\log_a(a^x) = x \text{ for all } x, \quad a^{\log_a x} = x \text{ for } x > 0.$$

The graph of $\log_a x$ is the reflection of the graph of a^x in $y = x$: domain $(0, \infty)$, range $\mathbb{R}$, passing through $(1, 0)$, with the y -axis as a vertical asymptote.

Example 4.7 (Domains with logarithms). The domain of $f(x) = \ln(x^2 - 4x)$ requires $x^2 - 4x > 0$, that is $x(x - 4) > 0$: $(-\infty, 0) \cup (4, \infty)$. The graph of $y = \log_2(x - 3) + 1$ is $\log_2 x$ shifted right 3 and up 1, with vertical asymptote $x = 3$.

4.3 LAWS OF LOGARITHMS

Theorem 4.8 (Laws of logarithms). For $a > 0$, $a \neq 1$, $x, y > 0$, and any real r :

$$\log_a(xy) = \log_a x + \log_a y, \quad \log_a \frac{x}{y} = \log_a x - \log_a y, \quad \log_a(x^r) = r \log_a x.$$

Proof. Let $u = \log_a x$ and $v = \log_a y$, so $a^u = x$ and $a^v = y$. Then $xy = a^u a^v = a^{u+v}$, so $\log_a(xy) = u + v$. Similarly $\frac{x}{y} = a^{u-v}$. And $x^r = (a^u)^r = a^{ru}$, so $\log_a(x^r) = ru$. ■

Theorem 4.9 (Change of base). For bases $a, b > 0$ different from 1, $\log_b x = \frac{\log_a x}{\log_a b}$. In particular $\log_b x = \frac{\ln x}{\ln b}$.

Proof. Let $y = \log_b x$, so $b^y = x$. Take $\log_a$ of both sides: $y \log_a b = \log_a x$. ■

Example 4.10 (Expanding and combining). (a) $\ln \frac{x^3 \sqrt{y}}{z^2} = 3 \ln x + \frac{1}{2} \ln y - 2 \ln z$. (b) $2 \log x - \frac{1}{3} \log(x + 1) + \log 5 = \log \frac{5x^2}{\sqrt[3]{x+1}}$. (c) $\log_7 25 = \frac{\ln 25}{\ln 7} \approx 1.654$.

COMMON ERROR (False laws). $\log_a(x + y) \neq \log_a x + \log_a y$; there is no law for the logarithm of a sum. Also $\frac{\log_a x}{\log_a y} \neq \log_a \frac{x}{y}$, and $(\log_a x)^r \neq r \log_a x$. Test any suspected law with numbers: $\log_{10}(10 + 10) = 1.301 \neq 2$.

4.4 EXPONENTIAL AND LOGARITHMIC EQUATIONS

Example 4.11 (Exponential equations). Solve (a) $3^{x+2} = 7$, (b) $e^{2x} - e^x - 6 = 0$, (c) $5^x = 2^{3x-1}$.

Solution. (a) Take $\ln$: $(x+2)\ln 3 = \ln 7$, so $x = \frac{\ln 7}{\ln 3} - 2 \approx -0.229$.

(b) A quadratic in e^x : $(e^x - 3)(e^x + 2) = 0$. Since $e^x > 0$, only $e^x = 3$: $x = \ln 3$.

(c) $x \ln 5 = (3x - 1) \ln 2$, so $x(\ln 5 - 3 \ln 2) = -\ln 2$ and $x = \frac{\ln 2}{3 \ln 2 - \ln 5} = \frac{\ln 2}{\ln(8/5)} \approx 1.475$.

Example 4.12 (Logarithmic equations). Solve $\log_2(x+2) + \log_2(x-1) = 2$.

Solution. Combine: $\log_2[(x+2)(x-1)] = 2$, so $(x+2)(x-1) = 4$, $x^2 + x - 6 = 0$, $x = 2$ or $x = -3$. Check: at $x = -3$, $\log_2(-1)$ is undefined, so -3 is extraneous. The solution is $x = 2$.

COMMON ERROR (Extraneous solutions from combining logs). Combining $\log_a u + \log_a v$ into $\log_a(uv)$ enlarges the domain: $uv > 0$ also when u and v are both negative. Always check solutions in the *original* equation.

4.5 MODELING

Theorem 4.13 (Exponential growth and decay). *A quantity that changes at a rate proportional to its size satisfies $A(t) = A_0 e^{kt}$, where $A_0 = A(0)$. It grows if $k > 0$ and decays if $k < 0$. The doubling time (for $k > 0$) is $\frac{\ln 2}{k}$ and the half-life (for $k < 0$) is $\frac{\ln 2}{|k|}$.*

The phrase “rate proportional to its size” is a statement about derivatives; Volume III proves that it forces the exponential form. Here we take the model as given.

Example 4.14 (Carbon dating). Carbon-14 has a half-life of 5730 years. A charcoal sample has 22% of the carbon-14 of living wood. Estimate its age.

Solution. $k = -\frac{\ln 2}{5730}$. Solve $e^{kt} = 0.22$: $t = \frac{\ln 0.22}{k} = 5730 \cdot \frac{\ln(1/0.22)}{\ln 2} \approx 12,520$ years.

Example 4.15 (Population). A bacteria culture has 500 cells at $t = 0$ and 8000 after 4 hours. Find the growth model and when the population reaches 50,000.

Solution. $8000 = 500e^{4k}$ gives $e^{4k} = 16$ and $k = \frac{\ln 16}{4} = \ln 2$: the culture doubles every hour, $N(t) = 500 \cdot 2^t$. Then $2^t = 100$ gives $t = \log_2 100 \approx 6.64$ hours.

4.5.1 LOGARITHMIC SCALES

When a quantity ranges over many orders of magnitude, its logarithm is easier to work with. The pH of a solution is $\text{pH} = -\log[\text{H}^+]$; the loudness of a sound in decibels is $\beta = 10 \log \frac{I}{I_0}$; and the Richter magnitude of an earthquake is $M = \log \frac{S}{S_0}$, where S is a standard amplitude. On each scale, adding a fixed amount corresponds to *multiplying* the underlying quantity by a fixed factor.

Example 4.16 (Comparing earthquakes). The 2011 Tōhoku earthquake had magnitude 9.1 and the 1989 Loma Prieta earthquake 6.9. The ratio of their amplitudes is $10^{9.1-6.9} = 10^{2.2} \approx 158$.

4.6 FURTHER MODELS

4.6.1 NEWTON'S LAW OF COOLING

An object at temperature T_0 placed in surroundings at constant temperature T_s cools (or warms) so that the *difference* $T - T_s$ decays exponentially:

$$T(t) = T_s + (T_0 - T_s)e^{-kt}, \quad k > 0.$$

Example 4.17 (Time of death). At 2:00 a.m. a body is found in a room kept at 68°F. Its temperature is 85°F, and one hour later it is 80°F. Assuming a normal body temperature of 98.6°F, estimate the time of death.

Solution. Measure t in hours from 2:00 a.m.: $T(t) = 68 + 17e^{-kt}$. From $T(1) = 80$, $e^{-k} = \frac{12}{17}$, so $k = \ln \frac{17}{12} \approx 0.3483$. Death occurred when $T = 98.6$: $17e^{-kt} = 30.6$, so $t = -\frac{\ln(30.6/17)}{k} \approx -1.69$. That is about 1 hour 41 minutes before 2:00 a.m., around 12:19 a.m.

4.6.2 LOGISTIC GROWTH

Unlimited exponential growth is unrealistic for populations, epidemics, or the spread of information, because resources or susceptible people run out. The *logistic model* has an S-shaped graph that levels off at a *carrying capacity* M :

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \quad A = \frac{M - P_0}{P_0}.$$

For small t it grows almost exponentially; it grows fastest when $P = \frac{M}{2}$; and $P(t) \rightarrow M$ as $t \rightarrow \infty$.

Example 4.18 (A rumor). In a school of 1000 students, 10 initially know a rumor, and 100 know it after 2 days. When will half the school know it?

Solution. $A = \frac{990}{10} = 99$. From $P(2) = 100$: $1 + 99e^{-2k} = 10$, so $e^{-2k} = \frac{1}{11}$ and $k = \frac{\ln 11}{2} \approx 1.199$. Half the school knows it when $99e^{-kt} = 1$, at $t = \frac{\ln 99}{k} \approx 3.83$ days.

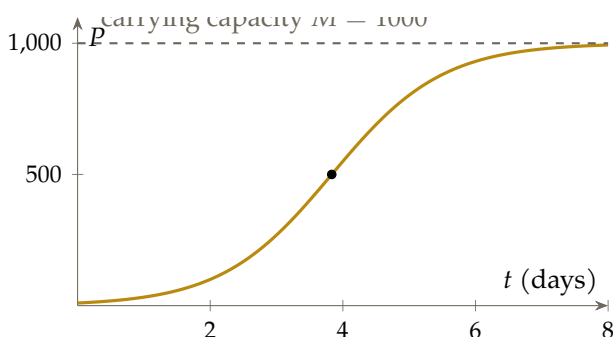


Figure 4.2: The logistic curve of Example 4.18. Growth is fastest at the inflection point $P = 500$.

4.6.3 STRAIGHTENING DATA WITH LOGARITHMS

Logarithms turn exponential and power relationships into straight lines, which is how scientists recognize them in data.

- If $y = Ce^{kx}$, then $\ln y = \ln C + kx$: plotting $\ln y$ against x gives a line of slope k .
- If $y = Cx^p$, then $\ln y = \ln C + p \ln x$: plotting $\ln y$ against $\ln x$ gives a line of slope p .

Example 4.19 (Identifying a power law). Data: $(1, 2)$, $(2, 16)$, $(4, 128)$. Is y exponential or a power of x ?

Solution. Ratios of y for equal steps in x are not constant ($2 \rightarrow 16$ over a step of 1, $16 \rightarrow 128$ over a step of 2), so it is not exponential. On a log–log scale the points are $(0, \ln 2)$, $(\ln 2, 4 \ln 2)$, $(2 \ln 2, 7 \ln 2)$, which lie on a line of slope $\frac{3 \ln 2}{\ln 2} = 3$. So $y = Cx^3$, and $C = 2$: $y = 2x^3$.

LOOKING AHEAD. In calculus, e^x is the unique function equal to its own derivative, and $\ln x$ is the function whose derivative is $\frac{1}{x}$. Volume III turns the reasoning of this chapter upside down: it *defines* $\ln x$ as the area under $y = \frac{1}{t}$ from 1 to x , and then proves every law in Section 4.3 from that definition. Exponential decay is also the shape of a quantum wavefunction inside a barrier, which is why particles can tunnel through walls they classically could not climb.

4.7 WORKED PROBLEMS

These problems combine the laws of logarithms with equation solving and modeling.

PROBLEM 4.1 Different bases

Solve $2^{x+1} = 3^{x-1}$ exactly, and approximate the answer.

Solution. Take $\ln$ of both sides: $(x+1) \ln 2 = (x-1) \ln 3$. Collect x : $x(\ln 3 - \ln 2) = \ln 2 + \ln 3$, so

$$x = \frac{\ln 6}{\ln(3/2)} \approx 4.419.$$

Check: $2^{5.419} \approx 42.8$ and $3^{3.419} \approx 42.8$.

PROBLEM 4.2 An extraneous logarithmic solution

Solve $\ln x + \ln(x-3) = \ln 10$.

Solution. Combine: $\ln[x(x-3)] = \ln 10$, so $x^2 - 3x - 10 = 0$ and $x = 5$ or $x = -2$. The original equation requires $x > 3$ (both logarithms defined), so $x = -2$ is extraneous. The solution is $x = 5$: $\ln 5 + \ln 2 = \ln 10$.

PROBLEM 4.3 Mixed bases

Solve $\log_3 x + \log_9 x = 6$.

Solution. Convert to base 3: $\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{1}{2} \log_3 x$. So $\frac{3}{2} \log_3 x = 6$, $\log_3 x = 4$, and $x = 81$. Check: $\log_3 81 = 4$ and $\log_9 81 = 2$.

PROBLEM 4.4 An inequality that flips

Solve $0.8^t < 0.1$.

Solution. Take $\ln$ (an increasing function, so the inequality is preserved): $t \ln 0.8 < \ln 0.1$. Now divide by $\ln 0.8$, which is *negative*, so the inequality reverses: $t > \frac{\ln 0.1}{\ln 0.8} \approx 10.32$. Interpretation: a quantity losing 20% per period falls below a tenth of its starting value after about 10.32 periods.

PROBLEM 4.5 The inverse of a logarithmic function

Find the inverse of $f(x) = \ln(2x - 4) + 1$, with domains and ranges.

Solution. f has domain $(2, \infty)$ and range $\mathbb{R}$. Solve $y = \ln(2x - 4) + 1$: $2x - 4 = e^{y-1}$, so $x = \frac{1}{2}e^{y-1} + 2$. Thus $f^{-1}(x) = \frac{1}{2}e^{x-1} + 2$, with domain $\mathbb{R}$ and range $(2, \infty)$. The vertical asymptote $x = 2$ of f becomes the horizontal asymptote $y = 2$ of f^{-1} .

PROBLEM 4.6 Comparing interest rates

Which pays more: 5.9% compounded monthly, or 6% compounded annually?

Solution. Compare the *effective annual rates*, the actual percentage growth in one year. Monthly: $(1 + \frac{0.059}{12})^{12} - 1 \approx 0.06062$, or 6.06%. Annually: 6.00%. The lower nominal rate pays more because of more frequent compounding.

PROBLEM 4.7 Decay from two measurements

A sample has 80 mg at $t = 2$ hours and 50 mg at $t = 6$ hours. Find the half-life and the initial amount.

Solution. With $A(t) = A_0 e^{-kt}$, divide the two data points: $\frac{50}{80} = e^{-4k}$, so $k = \frac{\ln 1.6}{4} \approx 0.1175$. Half-life = $\frac{\ln 2}{k} = \frac{4 \ln 2}{\ln 1.6} \approx 5.90$ hours. Then $A_0 = 80e^{2k} = 80\sqrt{1.6} \approx 101.2$ mg.

PROBLEM 4.8 An equation quadratic in e^x

Solve $e^x - e^{-x} = 3$.

Solution. Multiply by e^x : $e^{2x} - 3e^x - 1 = 0$. With $u = e^x$, $u = \frac{3 \pm \sqrt{13}}{2}$. Since $u > 0$, only the plus sign is allowed, and $x = \ln \frac{3 + \sqrt{13}}{2} \approx 1.195$. (The left side is $2 \sinh x$, and this computes $\operatorname{arsinh} \frac{3}{2}$.)

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- | | |
|---|---|
| D4.1 Evaluate $\log_2 32$. | D4.14 Solve $\ln x = -1$. |
| D4.2 Evaluate $\log_3 \frac{1}{9}$. | D4.15 Solve $\log x + \log 4 = 2$. |
| D4.3 Evaluate $\log 0.01$. | D4.16 Solve $4^x = 2^{x+3}$. |
| D4.4 Evaluate $\ln e^{-4}$. | D4.17 Domain of $\log(5 - x)$. |
| D4.5 Evaluate $e^{\ln 5}$. | D4.18 Asymptote of $y = e^x + 2$. |
| D4.6 Write $\log_5 x = 3$ in exponential form. | D4.19 Asymptote of $y = \ln(x + 3)$. |
| D4.7 Write $10^{-2} = 0.01$ in logarithmic form. | D4.20 Doubling time at continuous rate 5%. |
| D4.8 Expand $\log_2(8x^3)$. | D4.21 Value of \$1000 at 4% compounded continuously for 5 years. |
| D4.9 Combine $\ln a - 2 \ln b$. | D4.22 Half-life if $k = -0.1$ per year. |
| D4.10 Solve $3^x = 81$. | D4.23 $\log_4 7$ in terms of natural logs. |
| D4.11 Solve $5^{x-1} = 25$. | D4.24 Simplify $2^{\log_2 7 + 1}$. |
| D4.12 Solve $e^x = 10$. | |
| D4.13 Solve $\log_2(x - 1) = 3$. | |

ANSWERS TO SKILL DRILLS

D4.1 5. D4.2 -2. D4.3 -2. D4.4 -4. D4.5 5. D4.6 $5^3 = x$. D4.7 $\log 0.01 = -2$. D4.8 $3 + 3 \log_2 x$. D4.9 $\ln \frac{a}{b^2}$. D4.10 4. D4.11 3. D4.12 $\ln 10$. D4.13 9. D4.14 e^{-1} . D4.15 25. D4.16 3. D4.17 $(-\infty, 5)$. D4.18 $y = 2$. D4.19 $x = -3$. D4.20 $20 \ln 2 \approx 13.86$ years. D4.21 $\approx \$1221.40$. D4.22 $10 \ln 2 \approx 6.93$ years. D4.23 $\frac{\ln 7}{\ln 4}$. D4.24 14.

EXERCISES

COMPUTATION

- 4.1** Sketch $y = 3^{-x}$, $y = 2^x - 3$, and $y = -e^{x+1}$, giving the asymptote of each.
- 4.2** Find the exponential function $f(x) = Ca^x$ whose graph passes through $(1, 6)$ and $(3, 24)$.
- 4.3** \$2000 is invested at 4.5% interest. Find the balance after 8 years if interest is compounded quarterly, and if it is compounded continuously.
- 4.4** How long does it take an investment to double at 7% compounded continuously? At 7% compounded annually?
- 4.5** Evaluate without a calculator: $\log_3 81$, $\log_5 \frac{1}{125}$, $\log_{1/2} 8$, $\ln \sqrt{e}$, $10^{\log 7}$, $e^{2 \ln 3}$.
- 4.6** Find the domain of $f(x) = \log_3(9 - x^2)$ and of $g(x) = \ln(\ln x)$.
- 4.7** Expand: $\log \frac{x^2(x-1)}{\sqrt{x+1}}$.
- 4.8** Expand: $\ln \sqrt[3]{\frac{x^4 y}{z^5}}$.
- 4.9** Combine into one logarithm: $3 \ln x - 2 \ln(x^2 + 1) + \frac{1}{2} \ln(x - 1)$.
- 4.10** Combine: $\log_2 5 + 2 \log_2 3 - \log_2 15$, and simplify.
- 4.11** Use change of base to evaluate $\log_3 12$ and $\log_{0.5} 10$ to four decimals.
- 4.12** Solve $2^{3x-1} = 32$.
- 4.13** Solve $4^x = 7^{x-2}$.
- 4.14** Solve $e^{2x} - 5e^x + 6 = 0$.
- 4.15** Solve $\frac{10}{1 + e^{-x}} = 4$.
- 4.16** Solve $\log_3(x + 4) = 2$ and $\ln(2x - 1) = 3$.
- 4.17** Solve $\log x + \log(x - 15) = 2$.
- 4.18** Solve $\log_2(x + 3) - \log_2(x - 1) = 3$.
- 4.19** Solve $(\ln x)^2 - \ln(x^3) = 4$.
- 4.20** Solve $x^{\ln x} = e^4$, $x > 0$.

- 4.21** A radioactive isotope has a half-life of 30 years. How much of a 100-mg sample remains after 80 years? When will 10 mg remain?
- 4.22** A population grows from 1200 to 1800 in 5 years. Assuming exponential growth, find the population after 12 years and the doubling time.
- 4.23** A cup of coffee at 90°C cools in a 20°C room so that the temperature difference decays exponentially. After 10 minutes the coffee is at 60°C . When will it be at 40°C ?
- 4.24** The pH of lemon juice is 2.3 and that of milk is 6.6. How many times greater is the hydrogen ion concentration of lemon juice?
- 4.25** A sound of 80 dB is how many times as intense as a sound of 60 dB? What is the decibel level when two 70-dB sounds play together?

CONCEPTS AND PROOF

- 4.26** Prove that $\log_a \frac{1}{x} = -\log_a x$ and that $\log_a b \cdot \log_b a = 1$.
- 4.27** Prove that $a^{\log_b c} = c^{\log_b a}$ for positive a, b, c with $b \neq 1$.
- 4.28** Explain why $\log_a x$ is undefined for $x \leq 0$, and why the base $a = 1$ is excluded.
- 4.29** Show that $f(x) = \frac{e^x - e^{-x}}{2}$ is one-to-one and find its inverse. (This is $\sinh x$, and its inverse is $\operatorname{arsinh} x$.)
- 4.30** Prove the doubling-time formula: if $A = A_0 e^{kt}$ with $k > 0$, the time to double is $\frac{\ln 2}{k}$, independent of A_0 and of the starting time.
- 4.31** The “rule of 70” says that a quantity growing at $r\%$ per year doubles in about $\frac{70}{r}$ years. Explain where 70 comes from.
- 4.32** Show that $(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}$ for $n = 1, 2, 3, 10$ numerically, and explain what this suggests.
- 4.33** Which is larger, π^e or e^π ? Decide by comparing $\frac{\ln \pi}{\pi}$ with $\frac{\ln e}{e}$, and explain why that comparison decides it.
- 4.34** Show that $\log_2 3$ is irrational.
- 4.35** Explain why the graph of $y = \log_a x$ for $a > 1$ grows without bound but does so more slowly than any positive power x^p .

CHALLENGE

4.36 Solve $\log_2 x + \log_4 x + \log_8 x = 11$.

4.37 Solve $2^x + 2^{-x} = 3$.

4.38 Solve the system $\log_2 x + \log_2 y = 5$, $\log_2 x - \log_2 y = 1$.

4.39 Find all $x > 0$ with $x^{(x^2)} = (x^x)^2$.

4.40 The *logistic* model $P(t) = \frac{M}{1 + Ae^{-kt}}$ describes growth limited by a carrying capacity M . If $M = 1000$, $P(0) = 100$, and $P(2) = 200$, find A and k , and find when $P = 500$.



PART II

TRIGONOMETRY

Angles and the circular functions, their graphs and inverses,
identities and equations, and the solution of triangles.

CHAPTER 5

ANGLES AND THE TRIGONOMETRIC FUNCTIONS

LEARNING OBJECTIVES

- Measure angles in degrees and radians, and compute arc length, sector area, and angular speed.
- Define the six trigonometric functions on the unit circle, and evaluate them exactly at the special angles.
- Use reference angles and quadrant signs to evaluate trigonometric functions of any angle.
- Solve right triangles and applied problems with angles of elevation and depression.
- Use the fundamental identities to find all six function values from one.

Trigonometry began as the study of triangles: surveyors and astronomers needed to compute distances they could not measure directly. Its modern importance is different. The trigonometric functions are the fundamental *periodic* functions, and every repeating process, from a vibrating string to an alternating current to a quantum wave, is described by them. This chapter defines them in the way that serves both purposes: on the unit circle, with angles measured in radians.

5.1 ANGLES AND THEIR MEASURE

An angle is formed by rotating a ray (the *initial side*) about its endpoint to a final position (the *terminal side*). Counterclockwise rotation gives a positive angle, clockwise a negative one. An angle is in *standard position* when its vertex is at the origin and its initial side lies along the positive x -axis. Angles in standard position with the same terminal side are *coterminal*; they differ by a multiple of

a full revolution.

Definition 5.1 (Radian measure). If a circle of radius r is drawn with its center at the vertex of an angle, and the angle cuts off an arc of length s , the measure of the angle in *radians* is $\theta = \frac{s}{r}$.

A full revolution cuts off the whole circumference $2\pi r$, so it measures 2π radians. Hence

$$360^\circ = 2\pi \text{ rad}, \quad 1^\circ = \frac{\pi}{180} \text{ rad}, \quad 1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ.$$

A radian is a ratio of two lengths, so it has no units; that is why it is the natural measure in calculus.

Theorem 5.2 (Arc length, sector area, angular speed). In a circle of radius r , a central angle of θ radians cuts off an arc of length $s = r\theta$ and a sector of area $A = \frac{1}{2}r^2\theta$. An object moving around the circle through angle θ in time t has angular speed $\omega = \theta/t$ and linear speed $v = r\omega$.

Proof. The arc length formula is the definition of radian measure. The sector is the fraction $\frac{\theta}{2\pi}$ of the disk, so its area is $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$. Dividing $s = r\theta$ by t gives $v = r\omega$. ■

Example 5.3 (A bicycle wheel). A bicycle wheel of diameter 26 inches turns at 200 revolutions per minute. Find the bicycle's speed in miles per hour.

Solution. $\omega = 200 \cdot 2\pi = 400\pi$ rad/min, and $v = r\omega = 13 \cdot 400\pi = 5200\pi$ in/min. Converting: $\frac{5200\pi \cdot 60}{12 \cdot 5280} \approx 15.5$ mi/h.

COMMON ERROR (Degrees in formulas). $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ require θ in radians. A 60° angle in a circle of radius 10 cuts off an arc of length $10 \cdot \frac{\pi}{3} \approx 10.47$, not $10 \cdot 60 = 600$.

5.2 THE UNIT CIRCLE

Definition 5.4 (Trigonometric functions). Let t be a real number, and let $P(x, y)$ be the point reached by traveling a distance $|t|$ along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$, counterclockwise if $t > 0$ and clockwise if $t < 0$. (P is the *terminal point* of t ; equivalently, P is on the terminal side of the angle of t radians in standard position.) Define

$$\sin t = y, \quad \cos t = x, \quad \tan t = \frac{y}{x}, \quad \csc t = \frac{1}{y}, \quad \sec t = \frac{1}{x}, \quad \cot t = \frac{x}{y},$$

wherever the denominators are not zero.

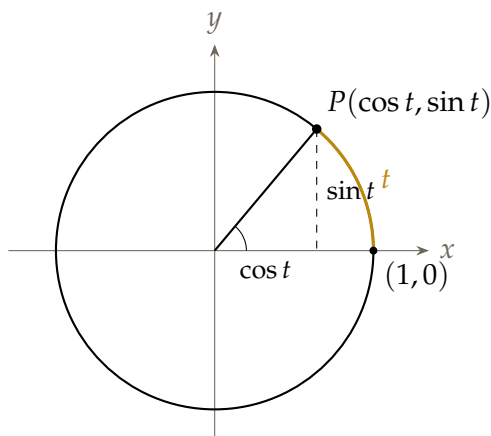


Figure 5.1: The terminal point of t on the unit circle. Its coordinates are $(\cos t, \sin t)$.

Theorem 5.5 (Values at the special angles).

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan t$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Proof. For $t = \frac{\pi}{4}$ the terminal point lies on $y = x$, so $x^2 + x^2 = 1$ and $x = y = \frac{\sqrt{2}}{2}$. For $t = \frac{\pi}{3}$, the points $(0, 0)$, $(1, 0)$, and P form a triangle with two sides of length 1 and the angle between them 60° , so it is equilateral; P lies above the midpoint

of the base, $x = \frac{1}{2}$, and $y = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$. The point for $\frac{\pi}{6}$ is the reflection of that for $\frac{\pi}{3}$ in $y = x$. ■

5.2.1 REFERENCE NUMBERS

The *reference number* $\bar{t}$ of t is the shortest distance along the circle from the terminal point of t to the x -axis. The values of the trigonometric functions at t equal the values at $\bar{t}$ up to sign, and the sign is determined by the quadrant of the terminal point. A useful mnemonic for the positive functions in quadrants I–IV is “All Students Take Calculus”: all, sine, tangent, cosine.

Example 5.6 (Any angle). Evaluate $\cos \frac{5\pi}{6}$, $\tan(-\frac{\pi}{3})$, $\sin \frac{19\pi}{4}$, and $\sec \frac{4\pi}{3}$.

Solution. $\frac{5\pi}{6}$ is in quadrant II with reference number $\frac{\pi}{6}$; cosine is negative there: $-\frac{\sqrt{3}}{2}$. $-\frac{\pi}{3}$ is in quadrant IV with reference $\frac{\pi}{3}$; tangent is negative: $-\sqrt{3}$. $\frac{19\pi}{4} = 4\pi + \frac{3\pi}{4}$, coterminal with $\frac{3\pi}{4}$ (quadrant II, reference $\frac{\pi}{4}$): $\frac{\sqrt{2}}{2}$. $\frac{4\pi}{3}$ is in quadrant III with reference $\frac{\pi}{3}$; $\cos \frac{4\pi}{3} = -\frac{1}{2}$, so $\sec \frac{4\pi}{3} = -2$.

5.3 FUNDAMENTAL IDENTITIES

Theorem 5.7 (Fundamental identities). For all t where the functions are defined:

Reciprocal: $\csc t = \frac{1}{\sin t}$, $\sec t = \frac{1}{\cos t}$, $\cot t = \frac{1}{\tan t}$, $\tan t = \frac{\sin t}{\cos t}$;

Pythagorean: $\sin^2 t + \cos^2 t = 1$, $\tan^2 t + 1 = \sec^2 t$, $1 + \cot^2 t = \csc^2 t$;

Even-odd: $\sin(-t) = -\sin t$, $\cos(-t) = \cos t$, $\tan(-t) = -\tan t$;

Periodicity: $\sin(t + 2\pi) = \sin t$, $\cos(t + 2\pi) = \cos t$, $\tan(t + \pi) = \tan t$.

Proof. $\sin^2 t + \cos^2 t = y^2 + x^2 = 1$ because P is on the unit circle; dividing by $\cos^2 t$ and by $\sin^2 t$ gives the other two. The terminal point of $-t$ is the reflection of that of t in the x -axis, which negates y and keeps x . Adding 2π goes once around the circle; adding π moves P to the diametrically opposite point $(-x, -y)$, which leaves y/x unchanged. ■

Example 5.8 (From one value to all six). If $\cos t = -\frac{3}{5}$ and t is in quadrant III, find the other five values.

Solution. $\sin^2 t = 1 - \frac{9}{25} = \frac{16}{25}$, and sine is negative in quadrant III: $\sin t =$

$-\frac{4}{5}$. Then $\tan t = \frac{4}{3}$, $\csc t = -\frac{5}{4}$, $\sec t = -\frac{5}{3}$, $\cot t = \frac{3}{4}$.

5.4 RIGHT TRIANGLE TRIGONOMETRY

For an acute angle θ in a right triangle, place θ at the origin with the adjacent leg on the x -axis. Scaling the triangle by $\frac{1}{\text{hypotenuse}}$ puts the far vertex on the unit circle, which gives the classical definitions:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}}.$$

The two definitions agree; the unit-circle version simply extends them to angles that are not acute.

Example 5.9 (Height of a tree). From a point 50 ft from the base of a tree, the angle of elevation to the top is 38° . How tall is the tree?

Solution. $\tan 38^\circ = \frac{h}{50}$, so $h = 50 \tan 38^\circ \approx 39.06$ ft.

Example 5.10 (Two angles of elevation). From a point on level ground, the angle of elevation of a mountain peak is 32° . After walking 1000 m directly toward the mountain, it is 40° . Find the height of the mountain.

Solution. Let h be the height and x the final distance to the point below the peak. Then $h = x \tan 40^\circ$ and $h = (x + 1000) \tan 32^\circ$. Equating, $x(\tan 40^\circ - \tan 32^\circ) = 1000 \tan 32^\circ$, so $x \approx 2916.8$ m and $h = x \tan 40^\circ \approx 2447.5$ m.

COMMON ERROR (Calculator mode). $\sin 30$ on a calculator in radian mode is -0.988 , not 0.5 . Before every computation with degrees, check the mode. In calculus, radian mode is always correct.

LOOKING AHEAD. Defining $\sin$ and $\cos$ through arc length on the unit circle is what makes the fundamental limit $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$ true. That limit, proved in Volume III by comparing a triangle, a sector, and a second triangle, is the source of the derivative formulas $\frac{d}{dt} \sin t = \cos t$ and $\frac{d}{dt} \cos t = -\sin t$. In degrees, every one of those formulas would carry a stray factor $\frac{\pi}{180}$.

5.5 WORKED PROBLEMS

These problems combine radian measure, reference angles, the fundamental identities, and right-triangle trigonometry.

PROBLEM 5.1 A swinging pendulum

A pendulum 80 cm long swings through an angle of 15° . How far does the bob travel?

Solution. Convert to radians: $15^\circ = \frac{\pi}{12}$. The bob moves along an arc of radius 80: $s = r\theta = 80 \cdot \frac{\pi}{12} = \frac{20\pi}{3} \approx 20.94$ cm.

PROBLEM 5.2 Large and negative angles

Evaluate $\sin\left(-\frac{13\pi}{6}\right) + \cos\frac{17\pi}{3}$ exactly.

Solution. Sine is odd: $\sin\left(-\frac{13\pi}{6}\right) = -\sin\frac{13\pi}{6} = -\sin\left(2\pi + \frac{\pi}{6}\right) = -\frac{1}{2}$. And $\frac{17\pi}{3} - 6\pi = -\frac{\pi}{3}$, so $\cos\frac{17\pi}{3} = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$. The sum is 0.

PROBLEM 5.3 All six from one

If $\tan \theta = -2$ and $\cos \theta > 0$, find the other five values.

Solution. Tangent negative and cosine positive puts θ in quadrant IV. Use a point on the terminal side: $(1, -2)$, at distance $r = \sqrt{5}$. Then $\sin \theta = -\frac{2}{\sqrt{5}}$, $\cos \theta = \frac{1}{\sqrt{5}}$, $\csc \theta = -\frac{\sqrt{5}}{2}$, $\sec \theta = \sqrt{5}$, $\cot \theta = -\frac{1}{2}$. Check: $\sin^2 \theta + \cos^2 \theta = \frac{4}{5} + \frac{1}{5} = 1$.

PROBLEM 5.4 Two angles of elevation, exactly

From point A the angle of elevation to the top of a tower is 30° . From B , 100 ft closer, it is 45° . Find the tower's height exactly.

Solution. Let x be the distance from B to the base and h the height. Then $h = x \tan 45^\circ = x$ and $h = (x + 100) \tan 30^\circ = \frac{x+100}{\sqrt{3}}$. So $x\sqrt{3} = x + 100$, and

$$h = x = \frac{100}{\sqrt{3} - 1} = 50(\sqrt{3} + 1) \approx 136.6 \text{ ft.}$$

PROBLEM 5.5 Speed due to Earth's rotation

How fast is a point at latitude 40°N moving because of Earth's rotation? Use radius 3960 mi and a 24-hour day.

Solution. The point travels around a circle of radius $3960 \cos 40^\circ \approx 3033.5$ mi (the distance from Earth's axis). In 24 hours it covers 2π times that, so its speed is $\frac{2\pi(3033.5)}{24} \approx 794$ mi/h.

PROBLEM 5.6 Simplifying with identities

Simplify $\frac{\sec \theta - \cos \theta}{\tan \theta}$.

Solution. $\sec \theta - \cos \theta = \frac{1 - \cos^2 \theta}{\cos \theta} = \frac{\sin^2 \theta}{\cos \theta}$. Dividing by $\tan \theta = \frac{\sin \theta}{\cos \theta}$ gives $\sin \theta$ (wherever the original expression is defined).

PROBLEM 5.7 Area of a circular segment

A chord in a circle of radius 10 subtends a central angle of $\frac{2\pi}{3}$. Find the area between the chord and the arc.

Solution. Segment = sector - triangle = $\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = 50 \cdot \frac{2\pi}{3} - 50 \cdot \frac{\sqrt{3}}{2} = \frac{100\pi}{3} - 25\sqrt{3} \approx 61.42$.

PROBLEM 5.8 Finding the angle

Find all θ in $[0, 2\pi)$ with $\sin \theta = -\frac{\sqrt{3}}{2}$ and $\tan \theta > 0$.

Solution. Sine negative: quadrants III or IV. Tangent positive: quadrants I or III. So quadrant III, with reference angle $\frac{\pi}{3}$: $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D5.1 Convert 210° to radians.

D5.5 Arc length: $r = 4$, $\theta = \frac{3\pi}{2}$.

D5.2 Convert $\frac{5\pi}{4}$ to degrees.

D5.6 Sector area: $r = 6$, $\theta = \frac{\pi}{3}$.

D5.3 Convert $-\frac{\pi}{9}$ to degrees.

D5.7 $\sin \frac{3\pi}{4}$.

D5.4 Coterminal angle in $[0, 2\pi)$ for $\frac{17\pi}{6}$.

D5.8 $\cos \frac{5\pi}{3}$.

- D5.9** $\tan \frac{7\pi}{6}$.
- D5.10** $\sec \pi$.
- D5.11** $\csc \frac{3\pi}{2}$.
- D5.12** $\cot \frac{\pi}{3}$.
- D5.13** $\sin \left(-\frac{\pi}{4}\right)$.
- D5.14** $\cos \frac{11\pi}{6}$.
- D5.15** Reference angle of $\frac{8\pi}{5}$.
- D5.16** Quadrant of θ if $\sin \theta < 0$,
 $\cos \theta > 0$.
- D5.17** $\cos \theta$ if $\sin \theta = \frac{4}{5}$, θ in quadrant II.
- D5.18** $\tan \theta$ if $\cos \theta = \frac{5}{13}$, θ in quadrant IV.
- D5.19** Simplify $\cos x \tan x$.
- D5.20** Simplify $\frac{1-\sin^2 x}{\cos x}$.
- D5.21** Opposite leg if hypotenuse 10, angle 30° .
- D5.22** Angle with $\tan \theta = 1$,
 $0 < \theta < \frac{\pi}{2}$.
- D5.23** Linear speed: $r = 2$ m, $\omega = 5$ rad/s.
- D5.24** $\sin^2 \frac{\pi}{7} + \cos^2 \frac{\pi}{7}$.

ANSWERS TO SKILL DRILLS

D5.1 $\frac{7\pi}{6}$. D5.2 225° . D5.3 -20° . D5.4 $\frac{5\pi}{6}$. D5.5 6π . D5.6 6π . D5.7 $\frac{\sqrt{2}}{2}$.
D5.8 $\frac{1}{2}$. D5.9 $\frac{\sqrt{3}}{3}$. D5.10 -1 . D5.11 -1 . D5.12 $\frac{\sqrt{3}}{3}$. D5.13 $-\frac{\sqrt{2}}{2}$. D5.14 $\frac{\sqrt{3}}{2}$.
D5.15 $\frac{2\pi}{5}$. D5.16 IV. D5.17 $-\frac{3}{5}$. D5.18 $-\frac{12}{5}$. D5.19 $\sin x$. D5.20 $\cos x$.
D5.21 5. D5.22 $\frac{\pi}{4}$. D5.23 10 m/s. D5.24 1.

EXERCISES

COMPUTATION

- 5.1** Convert to radians: 72° , -150° , 585° . Convert to degrees: $\frac{7\pi}{12}$, $-\frac{5\pi}{3}$, 2.
- 5.2** Find a positive and a negative angle coterminal with $\frac{11\pi}{4}$, and the coterminal angle in $[0, 2\pi)$.
- 5.3** Find the arc length cut off by a central angle of 135° in a circle of radius 8 cm.
- 5.4** A sector of a circle of radius 12 m has area 60 m^2 . Find its central angle.
- 5.5** Two cities on the same meridian have latitudes 40°N and 25°N . Using Earth's radius 3960 mi, find the distance between them.
- 5.6** A Ferris wheel of radius 25 m completes a revolution every 90 seconds. Find the angular and linear speed of a rider.

- 5.7** Find the terminal point on the unit circle for $t = \frac{7\pi}{6}$, $t = -\frac{3\pi}{4}$, $t = \frac{11\pi}{3}$.
- 5.8** Find the reference number for $\frac{5\pi}{7}$, $-\frac{7\pi}{5}$, and 5.7 .
- 5.9** Evaluate exactly: $\sin \frac{2\pi}{3}$, $\cos \frac{7\pi}{4}$, $\tan \frac{5\pi}{6}$, $\csc \left(-\frac{\pi}{6}\right)$.
- 5.10** Evaluate exactly: $\sec \frac{11\pi}{6}$, $\cot \left(-\frac{3\pi}{4}\right)$, $\sin \frac{25\pi}{6}$, $\cos(-7\pi)$.
- 5.11** Evaluate exactly: $\sin 225^\circ$, $\cos(-300^\circ)$, $\tan 480^\circ$.
- 5.12** If $\sin t = \frac{5}{13}$ and t is in quadrant II, find the other five trigonometric values.
- 5.13** If $\tan \theta = -\frac{8}{15}$ and $\sin \theta > 0$, find $\cos \theta$ and $\csc \theta$.
- 5.14** If $\sec t = 3$ and $\tan t < 0$, find $\sin t$ and $\cot t$.
- 5.15** Write $\tan t$ in terms of $\sin t$ when t is in quadrant IV.
- 5.16** Simplify $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t}$ and $(1 - \cos t)(1 + \cos t) \csc^2 t$.
- 5.17** Solve the right triangle with hypotenuse 20 and one acute angle 35° .
- 5.18** Solve the right triangle with legs 7 and 24.
- 5.19** A 30-ft ladder leans against a wall at an angle of 72° with the ground. How high does it reach?
- 5.20** From the top of a 200-ft lighthouse, the angle of depression to a boat is 11° . How far is the boat from the base of the lighthouse?
- 5.21** A kite string is 150 m long and makes an angle of 50° with the ground. How high is the kite?
- 5.22** From a window 30 ft above the street, the angle of elevation to the top of a building across the street is 40° and the angle of depression to its base is 25° . How tall is the building?
- 5.23** Find the exact area of a regular hexagon inscribed in a circle of radius r .
- 5.24** Find the length of a chord that subtends a central angle of θ in a circle of radius r .
- 5.25** The minute hand of a clock is 6 cm long. How far does its tip travel in 25 minutes? What area does it sweep?

CONCEPTS AND PROOF

- 5.26** Prove that $\tan^2 t + 1 = \sec^2 t$ and $1 + \cot^2 t = \csc^2 t$ from $\sin^2 t + \cos^2 t = 1$.

- 5.27** Prove that $\sin(t + \pi) = -\sin t$ and $\cos(t + \pi) = -\cos t$, and deduce that $\tan$ has period π .
- 5.28** Explain why $|\sin t| \leq 1$ and $|\cos t| \leq 1$ for all t , while $\tan t$ takes every real value.
- 5.29** Prove that $\sin \theta < \theta < \tan \theta$ for $0 < \theta < \frac{\pi}{2}$ by comparing areas of two triangles and a sector.
- 5.30** Show that a central angle of 1 radian cuts off an arc equal in length to the radius, and estimate its size in degrees.
- 5.31** Explain why $\sin\left(\frac{\pi}{2} - t\right) = \cos t$ using a reflection of the unit circle.
- 5.32** Show that the area of a triangle with sides a , b and included angle C is $\frac{1}{2}ab \sin C$.
- 5.33** Is $\sin(2t) = 2 \sin t$? Disprove with a specific value and explain why such a “rule” cannot hold.
- 5.34** Show that $\cos^4 t - \sin^4 t = \cos^2 t - \sin^2 t$.
- 5.35** Prove that $|\sin a - \sin b| \leq |a - b|$ for a, b in $\left[0, \frac{\pi}{2}\right]$ using the unit circle (the chord is shorter than the arc).

CHALLENGE

- 5.36** Find $\sin \frac{\pi}{12}$ and $\cos \frac{\pi}{12}$ exactly, using a 30° – 60° – 90° triangle extended by an isosceles triangle. (Compare with Chapter 7.)
- 5.37** Find the exact value of $\sin 18^\circ$ using a 36° – 72° – 72° isosceles triangle and similar triangles.
- 5.38** Earth’s circumference was first measured by Eratosthenes: at noon on the solstice the sun was directly overhead at Syene, and 7.2° from vertical at Alexandria, about 5000 stadia north. Find his estimate of the circumference.
- 5.39** A belt runs around two pulleys of radii 6 cm and 2 cm whose centers are 16 cm apart, without crossing. Find the length of the belt.
- 5.40** Prove that if $\sin t = \sin s$ and $\cos t = \cos s$, then $t - s$ is an integer multiple of 2π .

CHAPTER 6

TRIGONOMETRIC GRAPHS AND INVERSE FUNCTIONS

LEARNING OBJECTIVES

- Graph $y = a \sin(k(x - b)) + d$ and $y = a \cos(k(x - b)) + d$, identifying amplitude, period, phase shift, and vertical shift.
- Build sinusoidal models from data and from physical descriptions, including simple and damped harmonic motion.
- Graph the tangent, cotangent, secant, and cosecant functions with their asymptotes.
- Define the inverse sine, cosine, and tangent functions on restricted domains, and evaluate compositions exactly.

The trigonometric functions repeat, and that single property makes them the language of oscillation. This chapter studies their graphs, shows how stretching and shifting a sine curve models tides, sound, and springs, and then constructs the inverse trigonometric functions, which recover an angle from a ratio.

6.1 GRAPHS OF SINE AND COSINE

A function f is *periodic* with period $p > 0$ if $f(x + p) = f(x)$ for all x , and p is the smallest such number. By Theorem 5.7, sin and cos have period 2π . One cycle of $\sin x$ rises from 0 to 1 at $\frac{\pi}{2}$, falls to 0 at π and to -1 at $\frac{3\pi}{2}$, and returns to 0 at 2π . Since $\cos x = \sin(x + \frac{\pi}{2})$, the cosine graph is the sine graph shifted left by $\frac{\pi}{2}$.

Theorem 6.1 (Sinusoidal curves). For $k > 0$, the curves $y = a \sin k(x - b) + d$

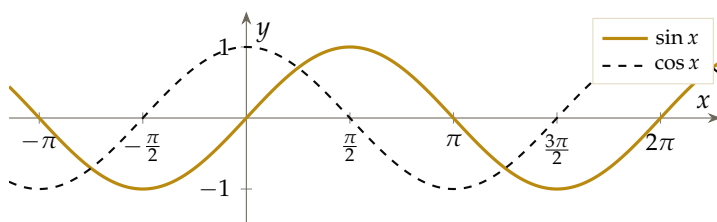


Figure 6.1: $y = \sin x$ and $y = \cos x$: the same wave, shifted by a quarter period.

and $y = a \cos k(x - b) + d$ have

amplitude $|a|$, period $\frac{2\pi}{k}$, phase shift b , midline $y = d$.

Proof. As x runs over one period $b \leq x \leq b + \frac{2\pi}{k}$, the quantity $k(x - b)$ runs over $[0, 2\pi]$, one full cycle of $\sin$. The factor a stretches the output to the interval $[-|a|, |a|]$, and adding d shifts it vertically. ■

Example 6.2 (Reading off the features). Graph $y = 3 \sin(2x - \frac{\pi}{2}) + 1$.

Solution. Factor the inside: $2x - \frac{\pi}{2} = 2(x - \frac{\pi}{4})$. Amplitude 3, period π , phase shift $\frac{\pi}{4}$ to the right, midline $y = 1$. One cycle starts on the midline at $x = \frac{\pi}{4}$, peaks at $y = 4$ when $x = \frac{\pi}{2}$, returns to the midline at $\frac{3\pi}{4}$, reaches its minimum -2 at π , and ends at $\frac{5\pi}{4}$.

COMMON ERROR (Phase shift without factoring). The phase shift of $\sin(2x - \frac{\pi}{2})$ is $\frac{\pi}{4}$, not $\frac{\pi}{2}$. Always write the inside as $k(x - b)$ before reading b .

6.1.1 SINUSOIDAL MODELS

Example 6.3 (Daylight hours). In a certain city the longest day (June 21, day 172) has 15.2 hours of daylight and the shortest (December 21) has 9.0. Model the daylight $D(t)$ on day t with a sinusoid of period 365 days.

Solution. Midline $\frac{15.2+9.0}{2} = 12.1$, amplitude $\frac{15.2-9.0}{2} = 3.1$, $k = \frac{2\pi}{365}$. A cosine peaks where its argument is 0, so $D(t) = 3.1 \cos \frac{2\pi}{365}(t - 172) + 12.1$.

Definition 6.4 (Simple harmonic motion). An object moves in *simple harmonic motion* if its displacement is $y = a \sin \omega t$ or $y = a \cos \omega t$. The *frequency* is $\frac{\omega}{2\pi}$ cycles per unit time, the reciprocal of the period. If the amplitude decays exponentially, as in $y = ae^{-ct} \sin \omega t$, the motion is *damped harmonic motion*.

Example 6.5 (A vibrating string). A guitar string vibrates at 440 cycles per second (the note A) with amplitude 2 mm. Its displacement is $y = 2 \sin(880\pi t)$, since $\omega = 2\pi \cdot 440$. If friction halves the amplitude every 0.5 seconds, the damped model is $y = 2e^{-ct} \sin(880\pi t)$ with $e^{-0.5c} = \frac{1}{2}$, so $c = 2 \ln 2 \approx 1.386$.

6.2 GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

Theorem 6.6 (Tangent and cotangent). $\tan x$ has period π and vertical asymptotes at $x = \frac{\pi}{2} + n\pi$; on each interval between asymptotes it increases from $-\infty$ to ∞ . $\cot x$ has period π , asymptotes at $x = n\pi$, and decreases on each interval.

The asymptotes come from the zeros of the denominators: $\tan x = \frac{\sin x}{\cos x}$ is undefined where $\cos x = 0$, and near such a point $|\tan x|$ is large. Similarly, $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$ have asymptotes at the zeros of $\cos$ and $\sin$; their graphs are U-shaped arcs touching the cosine and sine curves at their maxima and minima, with $|\sec x| \geq 1$ and $|\csc x| \geq 1$.

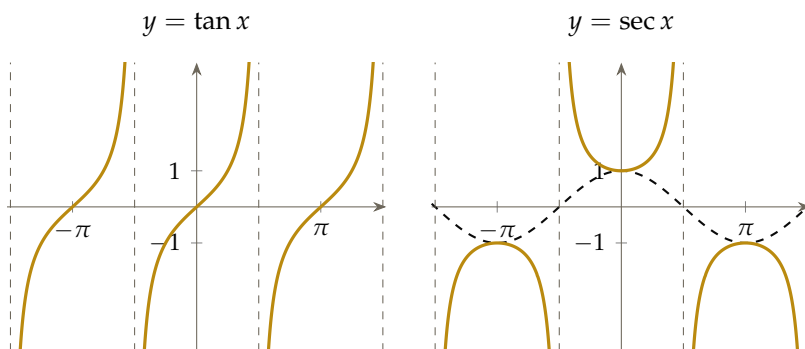


Figure 6.2: The tangent and secant functions, with vertical asymptotes where $\cos x = 0$.

6.3 INVERSE TRIGONOMETRIC FUNCTIONS

The trigonometric functions are not one-to-one, so to invert them we restrict each to an interval on which it is one-to-one and takes all its values (Example 2.19).

Definition 6.7 (Inverse sine, cosine, tangent).

$$\arcsin x = y \iff \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad (-1 \leq x \leq 1),$$

$$\arccos x = y \iff \cos y = x \text{ and } 0 \leq y \leq \pi \quad (-1 \leq x \leq 1),$$

$$\arctan x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2} \quad (x \in \mathbb{R}).$$

These are also written $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

The restricted intervals are chosen to contain the angles in $[0, \frac{\pi}{2}]$ and to be as natural as possible: $\arcsin$ and $\arctan$ return angles in quadrants I and IV, $\arccos$ in quadrants I and II.

Example 6.8 (Exact values). $\arcsin \frac{1}{2} = \frac{\pi}{6}$; $\arccos \left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$;
 $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$; $\arcsin(-1) = -\frac{\pi}{2}$.

Theorem 6.9 (Cancellation). $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$, but $\arcsin(\sin x) = x$ only for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Similarly for $\cos$ (on $[0, \pi]$) and $\tan$ (on $(-\frac{\pi}{2}, \frac{\pi}{2})$).

Example 6.10 (Cancellation outside the interval). $\arcsin(\sin \frac{5\pi}{6}) = \arcsin \frac{1}{2} = \frac{\pi}{6}$, not $\frac{5\pi}{6}$, because $\frac{5\pi}{6}$ is outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example 6.11 (Compositions by triangles). Find $\cos(\arctan x)$ and $\tan(\arccos \frac{3}{5})$.

Solution. Let $\theta = \arctan x$, so $\tan \theta = x$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, where $\cos \theta > 0$. Draw a right triangle with opposite side x and adjacent side 1; the hypotenuse is $\sqrt{1+x^2}$, so $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ (valid for all x , not only positive ones).

Let $\phi = \arccos \frac{3}{5}$, an angle in quadrant I with adjacent 3 and hypotenuse 5; the opposite side is 4, so $\tan \phi = \frac{4}{3}$.

COMMON ERROR ($\sin^{-1} x \neq \frac{1}{\sin x}$). $\sin^{-1} x$ is the inverse function, not the reciprocal. The reciprocal is $(\sin x)^{-1} = \csc x$. This is the one place where the notation $\sin^n x = (\sin x)^n$ breaks down, which is why this book prefers $\arcsin$.

6.4 MORE ON TRIGONOMETRIC GRAPHS

6.4.1 TRANSFORMED TANGENT AND SECANT GRAPHS

The rules of Theorem 6.1 apply to every trigonometric function, with one change: $\tan$ and $\cot$ have period π , so $y = a \tan k(x - b)$ has period $\frac{\pi}{k}$. To graph it, locate one pair of consecutive asymptotes, where $k(x - b) = \pm \frac{\pi}{2}$, then sketch one branch between them through the center $(b, 0)$.

Example 6.12 (A transformed tangent). Graph $y = 2 \tan \left(\frac{x}{2} - \frac{\pi}{4} \right)$.

Solution. Write the inside as $\frac{1}{2} \left(x - \frac{\pi}{2} \right)$: period $\frac{\pi}{1/2} = 2\pi$, shift $\frac{\pi}{2}$ to the right. Consecutive asymptotes are where $\frac{1}{2} \left(x - \frac{\pi}{2} \right) = \pm \frac{\pi}{2}$, at $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The branch crosses the axis at $x = \frac{\pi}{2}$ and passes through $(\pi, 2)$ and $(0, -2)$, where the inside equals $\pm \frac{\pi}{4}$.

6.4.2 SUMS AND PRODUCTS OF FUNCTIONS

Many signals are sums of sinusoids. When the frequencies are equal, the sum is a single sinusoid (Theorem 7.7). When they differ, the graph is more complex, but it is periodic whenever the ratio of the periods is rational, with period equal to their least common multiple.

Example 6.13 (Graphing a sum). The function $y = \sin x + \frac{1}{3} \sin 3x$ has period 2π (the least common multiple of 2π and $\frac{2\pi}{3}$). Adding the third harmonic flattens the tops of the sine wave; adding more odd harmonics, $\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots$, produces a curve that approaches a square wave. This is the first example of a *Fourier series*.

Example 6.14 (An envelope). For $y = x \sin x$, since $-1 \leq \sin x \leq 1$, the graph lies between the lines $y = x$ and $y = -x$ for $x \geq 0$, touching $y = x$ where $\sin x = 1$ and $y = -x$ where $\sin x = -1$. These lines are the *envelope*. Similarly $y = e^{-x/3} \cos 2x$ oscillates between $\pm e^{-x/3}$, which is how damped

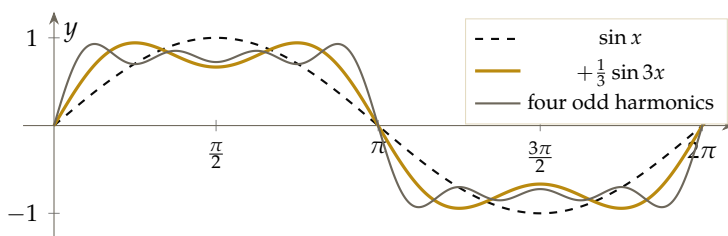


Figure 6.3: Adding odd harmonics to $\sin x$ builds an approximate square wave.

vibrations look.

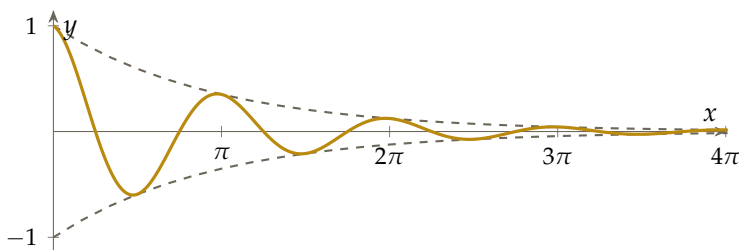


Figure 6.4: $y = e^{-x/3} \cos 2x$ between its envelope curves $y = \pm e^{-x/3}$.

6.4.3 FITTING A SINUSOID TO DATA

To model periodic data with $y = a \sin k(x - b) + d$ (or a cosine), read off four features: the midline d is the average of the maximum and minimum values; the amplitude a is half their difference; the period p is the time between successive maxima, giving $k = \frac{2\pi}{p}$; and the phase shift places a maximum (for cosine) or an upward midline crossing (for sine) at the right time.

Example 6.15 (Average monthly temperature). A city's average temperature is highest in July (month 7) at 84°F and lowest in January (month 1) at 30°F . Model it with a cosine.

Solution. $d = \frac{84+30}{2} = 57$, $a = \frac{84-30}{2} = 27$, period 12 months so $k = \frac{\pi}{6}$, and a maximum at $t = 7$: $T(t) = 27 \cos \frac{\pi}{6}(t - 7) + 57$. Check: $T(1) = 27 \cos(-\pi) + 57 = 30$. The model predicts $T(4) = 27 \cos(-\frac{\pi}{2}) + 57 = 57^\circ\text{F}$ in April.

LOOKING AHEAD. The inverse trigonometric functions have algebraic derivatives: $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ and $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$. Volume III proves these with the triangles of Example 6.11, and they make $\int \frac{dx}{1+x^2} = \arctan x + C$ one of the basic integrals. Sinusoids are the building blocks of Fourier analysis in Volume IV, where every periodic signal is written as a sum of sines and cosines of different frequencies.

6.5 WORKED PROBLEMS

These problems combine sinusoidal modeling, graph features, and inverse trigonometric functions. Each solution includes a check.

PROBLEM 6.1 A Ferris wheel

A Ferris wheel has radius 20 m, its center is 22 m above the ground, and it turns once every 4 minutes. A rider boards at the bottom at $t = 0$. Write the rider's height $h(t)$, and find how long during each revolution the rider is more than 32 m above the ground.

Solution. The height oscillates around the midline 22 with amplitude 20 and period 4, so $k = \frac{2\pi}{4} = \frac{\pi}{2}$. The rider starts at the minimum, which is the shape of $-\cos$: $h(t) = 22 - 20 \cos \frac{\pi t}{2}$. Check: $h(0) = 2$ (the bottom) and $h(2) = 42$ (the top).

$h(t) > 32$ means $\cos \frac{\pi t}{2} < -\frac{1}{2}$, so $\frac{\pi t}{2}$ lies in $(\frac{2\pi}{3}, \frac{4\pi}{3})$ and t in $(\frac{4}{3}, \frac{8}{3})$. The rider is above 32 m for $\frac{4}{3}$ minutes, 1 minute 20 seconds, of each revolution.

PROBLEM 6.2 Reading a graph

A sinusoid has a maximum at $(\frac{\pi}{8}, 3)$ and the next minimum at $(\frac{5\pi}{8}, -1)$. Find an equation for it.

Solution. Midline $\frac{3+(-1)}{2} = 1$; amplitude $\frac{3-(-1)}{2} = 2$. From a maximum to the next minimum is half a period, so the period is $2 \cdot \frac{4\pi}{8} = \pi$ and $k = 2$. A cosine peaks where its argument is 0, so put the shift at the maximum: $y = 2 \cos 2(x - \frac{\pi}{8}) + 1$. Check at $x = \frac{5\pi}{8}$: $2 \cos \pi + 1 = -1$.

PROBLEM 6.3 Inverse functions through triangles

Evaluate exactly: (a) $\sin(\arccos(-\frac{2}{3}))$; (b) $\tan(\arcsin(-\frac{3}{5}))$.

Solution. (a) Let $\theta = \arccos(-\frac{2}{3})$, so $\cos \theta = -\frac{2}{3}$ with θ in $[0, \pi]$, where sine is nonnegative. Then $\sin \theta = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$.

(b) Let $\phi = \arcsin(-\frac{3}{5})$, in $[-\frac{\pi}{2}, 0]$, quadrant IV, where cosine is positive: $\cos \phi = \frac{4}{5}$, so $\tan \phi = -\frac{3}{4}$.

In both parts the range of the inverse function, not a triangle alone, determines the sign.

PROBLEM 6.4 An equation with an inverse function

Solve $3 \arccos(2x - 1) = \pi$, and find the domain and range of $y = 2 \arcsin \frac{x}{3} - \frac{\pi}{2}$.

Solution. $\arccos(2x - 1) = \frac{\pi}{3}$ means $2x - 1 = \cos \frac{\pi}{3} = \frac{1}{2}$, so $x = \frac{3}{4}$. (This is valid because $\frac{\pi}{3}$ is in the range $[0, \pi]$ of $\arccos$.)

For $y = 2 \arcsin \frac{x}{3} - \frac{\pi}{2}$: the input of $\arcsin$ must satisfy $-1 \leq \frac{x}{3} \leq 1$, so the domain is $[-3, 3]$. $\arcsin$ outputs $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so y ranges over $[-\pi - \frac{\pi}{2}, \pi - \frac{\pi}{2}] = [-\frac{3\pi}{2}, \frac{\pi}{2}]$.

PROBLEM 6.5 Damping

A spring oscillates with $y = 5e^{-0.2t} \cos(\pi t)$ cm. What is its period, and after how long does the amplitude stay below 1 cm?

Solution. The cosine factor has period $\frac{2\pi}{\pi} = 2$ s. The amplitude is the envelope $5e^{-0.2t}$, which is below 1 when $e^{-0.2t} < \frac{1}{5}$, that is $t > \frac{\ln 5}{0.2} = 5 \ln 5 \approx 8.05$ s.

PROBLEM 6.6 The period of a sum

Find the period of $y = \sin \frac{\pi x}{3} + \cos \frac{\pi x}{4}$.

Solution. The periods are $\frac{2\pi}{\pi/3} = 6$ and $\frac{2\pi}{\pi/4} = 8$. The sum repeats when both terms do, at the least common multiple, 24. Check: in 24 units the first term completes 4 cycles and the second 3.

PROBLEM 6.7 The best view of a painting

A painting 10 ft tall hangs with its lower edge 2 ft above a viewer's eye level. Express the angle θ the painting subtends at the eye as a function of the viewer's distance x from the wall, and evaluate it at $x = 4$.

Solution. The top is 12 ft above eye level and the bottom 2 ft. The angles of elevation are $\arctan \frac{12}{x}$ and $\arctan \frac{2}{x}$, so $\theta(x) = \arctan \frac{12}{x} - \arctan \frac{2}{x}$. At $x = 4$: $\theta = \arctan 3 - \arctan \frac{1}{2}$. With the tangent subtraction formula of Chapter 7, $\tan \theta = \frac{3 - \frac{1}{3}}{1 + \frac{3}{2}} = 1$, so $\theta = \frac{\pi}{4}$, exactly 45° . (Calculus shows the angle is largest at $x = \sqrt{24} \approx 4.9$ ft.)

PROBLEM 6.8 Graphing a cosecant

Graph one period of $y = -2 \csc(2x + \frac{\pi}{2})$, giving its asymptotes and turning points.

Solution. Write the inside as $2(x + \frac{\pi}{4})$: period π , shift $\frac{\pi}{4}$ left. The asymptotes are where $\sin(2x + \frac{\pi}{2}) = 0$, that is $2x + \frac{\pi}{2} = n\pi$, or $x = -\frac{\pi}{4} + \frac{n\pi}{2}$. Since $\sin(2x + \frac{\pi}{2}) = \cos 2x$, the function is $y = -2 \sec 2x$: on $(-\frac{\pi}{4}, \frac{\pi}{4})$ it is a downward U with maximum -2 at $x = 0$, and on $(\frac{\pi}{4}, \frac{3\pi}{4})$ an upward U with minimum 2 at $x = \frac{\pi}{2}$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D6.1 Amplitude and period of
 $y = 4 \sin 3x$.

D6.2 Amplitude and period of
 $y = -\frac{1}{2} \cos \frac{x}{4}$.

D6.3 Period of $y = \tan 2x$.

D6.4 Phase shift of $y = \sin(x - \frac{\pi}{3})$.

D6.5 Phase shift of $y = \cos(2x + \pi)$.

D6.6 Midline of $y = 3 \sin x - 5$.

D6.7 Maximum value of
 $y = 2 \cos x + 1$.

D6.8 Minimum value of
 $y = -3 \sin 2x + 4$.

D6.9 Asymptotes of $y = \tan x$ on
 $[0, 2\pi]$.

D6.10 Range of $y = \sec x$.

D6.11 Period of $y = \csc \frac{\pi x}{2}$.

D6.12 $\arcsin 1$.

D6.13 $\arccos \frac{\sqrt{3}}{2}$.

D6.14 $\arctan(-1)$.

D6.15 $\arccos(-1)$.

D6.16 $\arcsin\left(-\frac{1}{2}\right)$.

D6.17 $\sin\left(\arcsin\frac{1}{3}\right)$.

D6.18 $\arccos\left(\cos\frac{5\pi}{3}\right)$.

D6.19 $\cos\left(\arctan\frac{3}{4}\right)$.

D6.20 $\tan\left(\arccos\frac{5}{13}\right)$.

D6.21 Domain of $\arcsin(x + 2)$.

D6.22 Frequency (cycles per second) of $y = \sin(10\pi t)$.

D6.23 Equation with amplitude 3, period π , no shift, sine shape.

D6.24 Solve $\arctan x = \frac{\pi}{3}$.

ANSWERS TO SKILL DRILLS

D6.1 amplitude 4, period $\frac{2\pi}{3}$. D6.2 amplitude $\frac{1}{2}$, period 8π . D6.3 $\frac{\pi}{2}$. D6.4 $\frac{\pi}{3}$ right. D6.5 $\frac{\pi}{2}$ left. D6.6 $y = -5$. D6.7 3. D6.8 1. D6.9 $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$. D6.10 $(-\infty, -1] \cup [1, \infty)$. D6.11 4. D6.12 $\frac{\pi}{2}$. D6.13 $\frac{\pi}{6}$. D6.14 $-\frac{\pi}{4}$. D6.15 π . D6.16 $-\frac{\pi}{6}$. D6.17 $\frac{1}{3}$. D6.18 $\frac{\pi}{3}$. D6.19 $\frac{4}{5}$. D6.20 $\frac{12}{5}$. D6.21 $[-3, -1]$. D6.22 5. D6.23 $y = 3 \sin 2x$. D6.24 $\sqrt{3}$.

EXERCISES

COMPUTATION

- 6.1** For $y = -4 \cos(3x + \pi) + 2$, find the amplitude, period, phase shift, and midline, and sketch one cycle.
- 6.2** Find the amplitude, period, and phase shift of $y = \frac{1}{2} \sin\left(\frac{x}{2} - \frac{\pi}{6}\right)$.
- 6.3** Write an equation $y = a \sin k(x - b)$ with amplitude 5, period $\frac{2\pi}{3}$, and phase shift $-\frac{\pi}{4}$.
- 6.4** A sinusoid has a maximum at $(1, 7)$ and the next minimum at $(4, -1)$. Find a cosine equation for it.
- 6.5** Find the period of $y = \sin(\pi x)$, $y = \tan(4x)$, and $y = \sec \frac{x}{3}$.
- 6.6** Sketch $y = \tan\left(x - \frac{\pi}{4}\right)$ on two periods, showing the asymptotes.
- 6.7** Sketch $y = 2 \csc x$ on $[0, 2\pi]$ together with $y = 2 \sin x$.
- 6.8** The water depth at a harbor varies from 2 m at low tide to 10 m at high tide, with high tides 12.4 hours apart. High tide is at 3:00. Write a model for the depth t hours after midnight, and find the depth at 9:00.
- 6.9** A mass on a spring oscillates with amplitude 6 cm and frequency 2 Hz, starting at maximum displacement at $t = 0$. Write its displacement.

- 6.10** A damped oscillator has $y = 4e^{-0.5t} \cos(2\pi t)$. When does the amplitude first fall below 1?
- 6.11** Evaluate exactly: $\arcsin \frac{\sqrt{3}}{2}$, $\arccos(-\frac{1}{2})$, $\arctan 1$, $\arccos 0$.
- 6.12** Evaluate exactly: $\arcsin(-\frac{\sqrt{2}}{2})$, $\arctan(-\frac{1}{\sqrt{3}})$, $\arccos 1$.
- 6.13** Evaluate: $\sin(\arcsin \frac{2}{5})$, $\arccos(\cos \frac{7\pi}{6})$, $\arctan(\tan \frac{3\pi}{4})$.
- 6.14** Evaluate exactly: $\cos(\arcsin \frac{5}{13})$, $\tan(\arccos(-\frac{4}{5}))$.
- 6.15** Evaluate exactly: $\sec(\arctan \frac{7}{24})$, $\csc(\arccos \frac{1}{3})$.
- 6.16** Write $\sin(\arccos x)$ and $\tan(\arcsin x)$ as algebraic expressions in x .
- 6.17** Use the identity $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$ to write $\cos(2 \arctan x)$ as an algebraic expression in x .
- 6.18** Find the domain and range of $y = \arcsin(2x - 1)$ and of $y = 3 \arctan x + \pi$.
- 6.19** Solve for x : $\arcsin(x - 2) = \frac{\pi}{6}$.
- 6.20** A 20-ft ladder leans against a wall with its foot x feet from the wall. Express the angle θ between the ladder and the ground as a function of x , and find θ when $x = 6$.

CONCEPTS AND PROOF

- 6.21** Prove that if f has period p , then $g(x) = f(kx)$ ($k > 0$) has period $\frac{p}{k}$.
- 6.22** Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$.
- 6.23** Prove that $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$. What is the value for $x < 0$?
- 6.24** Prove that $\arcsin(-x) = -\arcsin x$ but $\arccos(-x) = \pi - \arccos x$.
- 6.25** Explain why $\arccos$ is restricted to $[0, \pi]$ rather than $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- 6.26** Show that $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \phi)$ for some angle ϕ , and find ϕ . Use it to find the amplitude of $3 \sin x + 4 \cos x$.
- 6.27** Is $f(x) = \sin x + \sin(\sqrt{2}x)$ periodic? Explain informally.
- 6.28** Sketch $y = \arcsin(\sin x)$ on $[-2\pi, 2\pi]$. Explain its shape.
- 6.29** Show that the graph of $y = \sec x$ has a local minimum at $x = 0$ and a local maximum at $x = \pi$, using $|\cos x| \leq 1$.

6.30 Explain the difference between $\sin^{-1} x$, $(\sin x)^{-1}$, and $\sin(x^{-1})$.

CHALLENGE

6.31 Find the period of $y = \sin \frac{2x}{3} + \cos \frac{3x}{4}$.

6.32 Prove that $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$, using the tangent addition formula of Chapter 7 or a lattice-point diagram.

6.33 Solve $\arctan x + \arctan 2x = \frac{\pi}{4}$ for $x > 0$.

6.34 A rotating beacon 1 km from a straight shore turns at constant speed. Express the position of the beam on the shore as a function of the angle, and explain the vertical asymptotes in the graph.

6.35 Show that $\cos(\arcsin x) = \sqrt{1 - x^2}$ and $\sin(\arccos x) = \sqrt{1 - x^2}$, and explain why there is no sign ambiguity.

CHAPTER 7

TRIGONOMETRIC IDENTITIES AND EQUATIONS

LEARNING OBJECTIVES

- Verify trigonometric identities by working on one side with algebra and the fundamental identities.
- Prove the addition and subtraction formulas, and derive the double-angle, half-angle, and power-reduction formulas.
- Convert between products and sums of sines and cosines.
- Write $a \sin x + b \cos x$ as a single sinusoid.
- Solve trigonometric equations, finding every solution in an interval and the general solution.

An *identity* is an equation that holds for every value of the variable where both sides are defined. Trigonometric identities are tools for rewriting expressions into whatever form a problem needs: calculus uses them to integrate powers of sines, physics to combine waves, and this chapter to solve equations. Everything follows from one formula, the cosine of a difference, which we prove from the distance formula.

7.1 VERIFYING IDENTITIES

To verify an identity, transform one side, usually the more complicated, into the other. Do not treat the identity as an equation and manipulate both sides at once, since that assumes what is to be proved. Useful moves: rewrite everything in sines and cosines; combine fractions; factor; multiply by a conjugate; and use the Pythagorean identities.

Example 7.1 (Rewriting in sines and cosines). Verify $\frac{\tan x + \cot x}{\sec x \csc x} = 1$.

Solution. $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x}$, and $\sec x \csc x = \frac{1}{\sin x \cos x}$. Their quotient is 1.

Example 7.2 (Multiplying by a conjugate). Verify $\frac{\cos x}{1 - \sin x} = \sec x + \tan x$.

Solution. Multiply numerator and denominator by $1 + \sin x$:

$$\frac{\cos x(1 + \sin x)}{1 - \sin^2 x} = \frac{\cos x(1 + \sin x)}{\cos^2 x} = \frac{1 + \sin x}{\cos x} = \sec x + \tan x.$$

COMMON ERROR (Working on both sides). “Proving” $\sin x = -\sin x$ by squaring both sides to get $\sin^2 x = \sin^2 x$ shows why manipulating both sides is invalid: squaring is not reversible. Start from one side and arrive at the other.

7.2 ADDITION AND SUBTRACTION FORMULAS

Theorem 7.3 (Cosine of a difference). For all real s and t ,

$$\cos(s - t) = \cos s \cos t + \sin s \sin t.$$

Proof. The points $P = (\cos s, \sin s)$ and $Q = (\cos t, \sin t)$ lie on the unit circle, and the arc between them has length $|s - t|$. Rotating the circle so that Q moves to $(1, 0)$ carries P to $R = (\cos(s - t), \sin(s - t))$; rotation preserves distance, so $|PQ| = |R(1, 0)|$. Squaring both distances with the distance formula:

$$|PQ|^2 = (\cos s - \cos t)^2 + (\sin s - \sin t)^2 = 2 - 2(\cos s \cos t + \sin s \sin t),$$

$$|R(1, 0)|^2 = (\cos(s - t) - 1)^2 + \sin^2(s - t) = 2 - 2\cos(s - t),$$

using $\sin^2 + \cos^2 = 1$ in both. Equating gives the formula. ■

Theorem 7.4 (Addition and subtraction formulas).

$$\sin(s \pm t) = \sin s \cos t \pm \cos s \sin t, \quad \cos(s \pm t) = \cos s \cos t \mp \sin s \sin t,$$

$$\tan(s \pm t) = \frac{\tan s \pm \tan t}{1 \mp \tan s \tan t}.$$

Proof. Replace t by $-t$ in Theorem 7.3 to get $\cos(s + t)$. The cofunction identity $\sin u = \cos(\frac{\pi}{2} - u)$ (the case $s = \frac{\pi}{2}$ of Theorem 7.3) gives $\sin(s + t) = \cos((\frac{\pi}{2} - s) - t) = \cos(\frac{\pi}{2} - s) \cos t + \sin(\frac{\pi}{2} - s) \sin t = \sin s \cos t + \cos s \sin t$. For tangent, divide the formulas for sine and cosine and then divide numerator and denominator by $\cos s \cos t$. ■

Example 7.5 (Exact values). $\sin \frac{\pi}{12} = \sin(\frac{\pi}{3} - \frac{\pi}{4}) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$, and $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$.

Example 7.6 (Combining inverse functions). Evaluate $\sin(\arccos \frac{1}{2} + \arctan 2)$.

Solution. Let $\alpha = \arccos \frac{1}{2} = \frac{\pi}{3}$ and $\beta = \arctan 2$, with $\sin \beta = \frac{2}{\sqrt{5}}$, $\cos \beta = \frac{1}{\sqrt{5}}$. Then $\sin(\alpha + \beta) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{5}} + \frac{1}{2} \cdot \frac{2}{\sqrt{5}} = \frac{\sqrt{3}+2}{2\sqrt{5}}$.

7.2.1 SUMS OF SINES AND COSINES

Theorem 7.7 (Single sinusoid). For $(a, b) \neq (0, 0)$, $a \sin x + b \cos x = r \sin(x + \phi)$, where $r = \sqrt{a^2 + b^2}$ and ϕ satisfies $\cos \phi = \frac{a}{r}$, $\sin \phi = \frac{b}{r}$.

Proof. $r \sin(x + \phi) = r \sin x \cos \phi + r \cos x \sin \phi = a \sin x + b \cos x$. ■

For example, $\sin x + \sqrt{3} \cos x = 2 \sin(x + \frac{\pi}{3})$: the sum of two waves of the same frequency is a single wave of the same frequency.

7.3 DOUBLE-ANGLE, HALF-ANGLE, AND PRODUCT--SUM FORMULAS

Theorem 7.8 (Double-angle formulas).

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x, & \tan 2x &= \frac{2 \tan x}{1 - \tan^2 x}, \\ \cos 2x &= \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x. \end{aligned}$$

Proof. Set $s = t = x$ in Theorem 7.4; the alternate forms of $\cos 2x$ use $\sin^2 x + \cos^2 x = 1$. ■

Theorem 7.9 (Power-reduction and half-angle formulas).

$$\begin{aligned}\sin^2 x &= \frac{1 - \cos 2x}{2}, & \cos^2 x &= \frac{1 + \cos 2x}{2}, \\ \sin \frac{u}{2} &= \pm \sqrt{\frac{1 - \cos u}{2}}, & \cos \frac{u}{2} &= \pm \sqrt{\frac{1 + \cos u}{2}}, \\ \tan \frac{u}{2} &= \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u},\end{aligned}$$

where the sign is determined by the quadrant of $\frac{u}{2}$.

Proof. Solve $\cos 2x = 1 - 2\sin^2 x$ and $\cos 2x = 2\cos^2 x - 1$ for $\sin^2 x$ and $\cos^2 x$, then put $x = \frac{u}{2}$. For tangent, $\tan \frac{u}{2} = \frac{\sin(u/2)}{\cos(u/2)} = \frac{2\sin^2(u/2)}{2\sin(u/2)\cos(u/2)} = \frac{1 - \cos u}{\sin u}$. ■

Example 7.10 (Using double and half angles). If $\sin x = \frac{3}{5}$ with x in quadrant II, find $\sin 2x$, $\cos 2x$, and $\cos \frac{x}{2}$.

Solution. $\cos x = -\frac{4}{5}$. Then $\sin 2x = 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5}\right) = -\frac{24}{25}$ and $\cos 2x = 1 - 2 \cdot \frac{9}{25} = \frac{7}{25}$. Since $\frac{\pi}{2} < x < \pi$, $\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$, so $\cos \frac{x}{2} > 0$: $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}} = \frac{1}{\sqrt{10}}$.

Example 7.11 (A triple-angle formula). $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x = 4\cos^3 x - 3\cos x$.

Theorem 7.12 (Product-to-sum and sum-to-product formulas).

$$\begin{aligned}\sin u \cos v &= \frac{1}{2}[\sin(u + v) + \sin(u - v)], & \sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}, \\ \cos u \cos v &= \frac{1}{2}[\cos(u + v) + \cos(u - v)], & \cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}, \\ \sin u \sin v &= \frac{1}{2}[\cos(u - v) - \cos(u + v)], & \cos x - \cos y &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}.\end{aligned}$$

Proof. Add or subtract the addition formulas: $\sin(u + v) + \sin(u - v) = 2\sin u \cos v$, and similarly for the others. For the sum-to-product formulas, put $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$. ■

When two musical tones of nearly equal frequency sound together, $\sin at + \sin bt = 2\sin \frac{(a+b)t}{2} \cos \frac{(a-b)t}{2}$: a tone at the average frequency whose loudness

pulses slowly at the difference frequency. Musicians hear these *beats* when tuning.

7.4 TRIGONOMETRIC EQUATIONS

Because the trigonometric functions are periodic, an equation that has one solution typically has infinitely many. The strategy: find all solutions in one period, then add multiples of the period.

Example 7.13 (A basic equation). Solve $2 \sin x + 1 = 0$.

Solution. $\sin x = -\frac{1}{2}$. In $[0, 2\pi)$, sine is negative in quadrants III and IV with reference number $\frac{\pi}{6}$: $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. The general solution is $x = \frac{7\pi}{6} + 2k\pi$ or $x = \frac{11\pi}{6} + 2k\pi$, k an integer.

Example 7.14 (Quadratic in form). Solve $2 \cos^2 x + \sin x = 1$ on $[0, 2\pi)$.

Solution. Replace $\cos^2 x$ by $1 - \sin^2 x$: $2 - 2 \sin^2 x + \sin x = 1$, so $2 \sin^2 x - \sin x - 1 = 0$, $(2 \sin x + 1)(\sin x - 1) = 0$. Then $\sin x = 1$ gives $x = \frac{\pi}{2}$, and $\sin x = -\frac{1}{2}$ gives $x = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Example 7.15 (A multiple angle). Solve $\tan 3x = 1$ on $[0, 2\pi)$.

Solution. $3x = \frac{\pi}{4} + k\pi$, so $x = \frac{\pi}{12} + \frac{k\pi}{3}$. For x in $[0, 2\pi)$ take $k = 0, 1, \dots, 5$: $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$. Because $3x$ runs over $[0, 6\pi)$, there are three times as many solutions as for $\tan x = 1$.

Example 7.16 (Using an identity). Solve $\sin 2x = \cos x$ on $[0, 2\pi)$.

Solution. $2 \sin x \cos x - \cos x = 0$, so $\cos x(2 \sin x - 1) = 0$. $\cos x = 0$: $x = \frac{\pi}{2}, \frac{3\pi}{2}$. $\sin x = \frac{1}{2}$: $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

COMMON ERROR (Dividing by a trigonometric function). In Example 7.16, dividing by $\cos x$ loses the two solutions where $\cos x = 0$. Factor instead. Similarly, squaring both sides (to use $\sin^2 + \cos^2 = 1$) can introduce extraneous solutions that must be checked.

Example 7.17 (Squaring, with a check). Solve $\cos x + 1 = \sin x$ on $[0, 2\pi)$.

Solution. Squaring: $\cos^2 x + 2 \cos x + 1 = \sin^2 x = 1 - \cos^2 x$, so $2 \cos^2 x +$

$2 \cos x = 0$ and $\cos x = 0$ or $\cos x = -1$. Candidates: $\frac{\pi}{2}, \frac{3\pi}{2}, \pi$. Checking: $\frac{\pi}{2}$ gives $1 = 1$; π gives $0 = 0$; $\frac{3\pi}{2}$ gives $1 = -1$, false. Solutions: $x = \frac{\pi}{2}$ and $x = \pi$.

LOOKING AHEAD. The power-reduction formulas are exactly what calculus needs to integrate $\sin^2 x$ and $\cos^2 x$, and the product-to-sum formulas give the *orthogonality relations* $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$ for $m \neq n$, the foundation of Fourier series. In Volume IV, Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ turns every identity in this chapter into a consequence of the rule $e^{i(a+b)} = e^{ia}e^{ib}$.

7.5 WORKED PROBLEMS

These problems combine identities with equation solving. Notice how often the first step is choosing which identity to use.

PROBLEM 7.1 Verifying with double angles

Verify that $\frac{1 - \cos 2x}{\sin 2x} = \tan x$.

Solution. Use the forms of the double-angle formulas that remove the constant: $1 - \cos 2x = 2 \sin^2 x$ and $\sin 2x = 2 \sin x \cos x$. Then $\frac{2 \sin^2 x}{2 \sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$. The identity holds wherever $\sin 2x \neq 0$.

PROBLEM 7.2 An equation solved by combining

Solve $\sin 2x + \sqrt{3} \cos 2x = 1$ on $[0, 2\pi)$.

Solution. By Theorem 7.7 with $a = 1$, $b = \sqrt{3}$: $r = 2$ and $\phi = \frac{\pi}{3}$, so the equation is $2 \sin(2x + \frac{\pi}{3}) = 1$, that is $\sin(2x + \frac{\pi}{3}) = \frac{1}{2}$. Then $2x + \frac{\pi}{3} = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$, so $x = -\frac{\pi}{12} + k\pi$ or $x = \frac{\pi}{4} + k\pi$. In $[0, 2\pi)$:

$$x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{23\pi}{12}.$$

Check $x = \frac{\pi}{4}$: $\sin \frac{\pi}{2} + \sqrt{3} \cos \frac{\pi}{2} = 1$.

PROBLEM 7.3 A difference of inverse functions

Evaluate $\cos(\arctan 2 - \arcsin \frac{3}{5})$ exactly.

Solution. Let $A = \arctan 2$: from a triangle with legs 1 and 2, $\cos A = \frac{1}{\sqrt{5}}$ and $\sin A = \frac{2}{\sqrt{5}}$. Let $B = \arcsin \frac{3}{5}$: $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$. Then

$$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{4}{5\sqrt{5}} + \frac{6}{5\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

PROBLEM 7.4 Half angles with a sign decision

If $\tan \theta = -\frac{4}{3}$ and θ is in quadrant II, find $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$.

Solution. In quadrant II with $\tan \theta = -\frac{4}{3}$: $\cos \theta = -\frac{3}{5}$, $\sin \theta = \frac{4}{5}$. Since $\frac{\pi}{2} < \theta < \pi$, $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$, where sine and cosine are positive:

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}, \quad \cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \frac{1}{\sqrt{5}}, \quad \tan \frac{\theta}{2} = 2.$$

Check with the other formula: $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{4/5}{1 - 3/5} = \frac{4/5}{2/5} = 2.$

PROBLEM 7.5 Sum to product

Solve $\cos 3x = \cos x$ on $[0, 2\pi)$.

Solution. $\cos 3x - \cos x = -2 \sin 2x \sin x = 0$. So $\sin x = 0$ ($x = 0, \pi$) or $\sin 2x = 0$ ($x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$). Together: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

PROBLEM 7.6 Factoring a difference of fourth powers

Solve $\cos^4 x - \sin^4 x = \frac{1}{2}$ on $[0, 2\pi)$.

Solution. Factor: $(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = \cos 2x \cdot 1$. So $\cos 2x = \frac{1}{2}$, $2x = \pm \frac{\pi}{3} + 2k\pi$, $x = \pm \frac{\pi}{6} + k\pi$. In $[0, 2\pi)$: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

PROBLEM 7.7 Beats

Two tuning forks produce $y_1 = \sin(880\pi t)$ and $y_2 = \sin(888\pi t)$. Write the combined sound as a product and find how often its loudness pulses.

Solution. By sum-to-product, $y_1 + y_2 = 2 \sin(884\pi t) \cos(4\pi t)$. The first factor is a tone at 442 Hz, the average frequency. The second, $\cos(4\pi t)$, has frequency 2 Hz, and the loudness depends on its absolute value, which

peaks twice per cycle: the sound pulses 4 times per second, the difference $444 - 440$ of the frequencies.

PROBLEM 7.8 A maximum without calculus

Find the maximum value of $f(x) = 3 \sin x + 4 \cos x$ and a value of x where it occurs.

Solution. $f(x) = 5 \sin(x + \phi)$ with $\cos \phi = \frac{3}{5}$, $\sin \phi = \frac{4}{5}$, so $\phi = \arctan \frac{4}{3}$. The maximum is 5, when $x + \phi = \frac{\pi}{2}$, that is $x = \frac{\pi}{2} - \arctan \frac{4}{3} = \arctan \frac{3}{4} \approx 0.6435$. Check: $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$ there, and $3 \cdot \frac{3}{5} + 4 \cdot \frac{4}{5} = 5$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D7.1 Simplify $\sin^2 x + \cos^2 x + \tan^2 x$.

D7.2 Simplify $\frac{\sin x}{\tan x}$.

D7.3 Simplify $\sec^2 x - 1$.

D7.4 $\cos(a + b)$ if $\cos a = \frac{1}{2}$,
 $\sin a = \frac{\sqrt{3}}{2}$, $\cos b = 0$, $\sin b = 1$.

D7.5 Exact value of $\sin 75^\circ$.

D7.6 Exact value of $\cos 15^\circ$.

D7.7 Exact value of $\tan 105^\circ$.

D7.8 Simplify $\sin 3x \cos x - \cos 3x \sin x$.

D7.9 $\sin 2x$ if $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$.

D7.10 $\cos 2x$ if $\sin x = \frac{1}{3}$.

D7.11 Simplify $2 \cos^2 x - 1$.

D7.12 Simplify $\frac{2 \tan x}{1 - \tan^2 x}$.

D7.13 Exact value of $\sin \frac{\pi}{8}$.

D7.14 Write $\cos^2 x$ without a square.

D7.15 Write $\sin x \cos x$ as a single sine.

D7.16 Write $\sin 3x \cos x$ as a sum.

D7.17 Solve $\sin x = 0$ on $[0, 2\pi)$.

D7.18 Solve $\cos x = -\frac{\sqrt{2}}{2}$ on $[0, 2\pi)$.

D7.19 Solve $\tan x = \sqrt{3}$ on $[0, 2\pi)$.

D7.20 Solve $2 \sin x \cos x = 0$ on $[0, 2\pi)$.

D7.21 Solve $\sin 2x = 1$ on $[0, 2\pi)$.

D7.22 Solve $\cos^2 x = \frac{1}{4}$ on $[0, \pi]$.

D7.23 General solution of $\cos x = 1$.

D7.24 Amplitude of $\sin x + \cos x$.

ANSWERS TO SKILL DRILLS

D7.1 $1 + \tan^2 x = \sec^2 x$. D7.2 $\cos x$. D7.3 $\tan^2 x$. D7.4 $-\frac{\sqrt{3}}{2}$. D7.5 $\frac{\sqrt{6}+\sqrt{2}}{4}$.
D7.6 $\frac{\sqrt{6}+\sqrt{2}}{4}$. D7.7 $-2 - \sqrt{3}$. D7.8 $\sin 2x$. D7.9 $\frac{24}{25}$. D7.10 $\frac{7}{9}$. D7.11 $\cos 2x$.
D7.12 $\tan 2x$. D7.13 $\frac{\sqrt{2}-\sqrt{2}}{2}$. D7.14 $\frac{1+\cos 2x}{2}$. D7.15 $\frac{1}{2} \sin 2x$. D7.16
 $\frac{1}{2}(\sin 4x + \sin 2x)$. D7.17 $0, \pi$. D7.18 $\frac{3\pi}{4}, \frac{5\pi}{4}$. D7.19 $\frac{\pi}{3}, \frac{4\pi}{3}$. D7.20 $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
D7.21 $\frac{\pi}{4}, \frac{5\pi}{4}$. D7.22 $\frac{\pi}{3}, \frac{2\pi}{3}$. D7.23 $x = 2k\pi$. D7.24 $\sqrt{2}$.

EXERCISES

COMPUTATION

7.1 Verify $\sin x \cot x = \cos x$ and $\frac{\sec x}{\csc x} = \tan x$.

7.2 Verify $(\sin x + \cos x)^2 = 1 + \sin 2x$.

7.3 Verify $\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} = 2 \sec^2 x$.

7.4 Verify $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2 \csc x$.

7.5 Verify $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$.

7.6 Find the exact values of $\cos 105^\circ$, $\sin \frac{7\pi}{12}$, and $\tan \frac{\pi}{12}$.

7.7 Simplify $\sin \frac{\pi}{5} \cos \frac{3\pi}{10} + \cos \frac{\pi}{5} \sin \frac{3\pi}{10}$.

7.8 If $\sin s = \frac{3}{5}$ (s in quadrant I) and $\cos t = -\frac{5}{13}$ (t in quadrant III), find $\sin(s + t)$, $\cos(s - t)$, and the quadrant of $s + t$.

7.9 Write $\sin x - \cos x$ and $5 \sin x + 12 \cos x$ in the form $r \sin(x + \phi)$.

7.10 If $\tan x = -\frac{3}{4}$ with x in quadrant IV, find $\sin 2x$, $\cos 2x$, and $\tan 2x$.

7.11 Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, $\tan \frac{x}{2}$ if $\cos x = -\frac{7}{9}$ and $\pi < x < \frac{3\pi}{2}$.

7.12 Find the exact value of $\cos \frac{\pi}{8}$ and of $\tan 22.5^\circ$.

7.13 Express $\cos^4 x$ in terms of first powers of cosines of multiples of x .

7.14 Write $\sin 5x \sin 2x$ and $\cos 3x \cos x$ as sums.

7.15 Write $\sin 7x + \sin 3x$ and $\cos 5x - \cos x$ as products.

7.16 Solve on $[0, 2\pi)$: $2 \cos x - \sqrt{3} = 0$.

7.17 Solve on $[0, 2\pi)$: $\tan^2 x = 3$.

7.18 Solve on $[0, 2\pi)$: $2\sin^2 x - 3\sin x + 1 = 0$.

7.19 Solve on $[0, 2\pi)$: $\sin 2x = \sin x$.

7.20 Solve on $[0, 2\pi)$: $\cos 2x + 3\cos x + 2 = 0$.

7.21 Solve on $[0, 2\pi)$: $2\cos 3x = 1$.

7.22 Solve on $[0, 2\pi)$: $\sin x + \cos x = 1$.

7.23 Solve on $[0, 2\pi)$: $\sin 3x + \sin x = 0$.

7.24 Find the general solution of $\sec^2 x - 2\tan x = 4$.

7.25 Solve $\sqrt{3}\sin x + \cos x = 1$ on $[0, 2\pi)$ by first writing the left side as a single sinusoid.

CONCEPTS AND PROOF

7.26 Prove $\cos(s + t)\cos(s - t) = \cos^2 s - \sin^2 t$.

7.27 Prove $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$.

7.28 Prove $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$, and find where both sides are defined.

7.29 Derive $\sin 3x = 3\sin x - 4\sin^3 x$.

7.30 Prove that if $A + B + C = \pi$, then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

7.31 Explain why $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ implies that the sum of two equal-amplitude waves of nearby frequencies produces beats.

7.32 Show that the angle θ between two lines with slopes m_1, m_2 satisfies $\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$.

7.33 Derive the half-angle substitution $\sin x = \frac{2t}{1 + t^2}$, $\cos x = \frac{1 - t^2}{1 + t^2}$, where $t = \tan \frac{x}{2}$.

7.34 Explain the difference between the equations $\sin x = \frac{1}{2}$ and the identity $\sin(-x) = -\sin x$.

7.35 How many solutions does $\sin(nx) = 0$ have in $[0, 2\pi)$, for a positive integer n ?

CHALLENGE

7.36 Evaluate $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}$ exactly. *Hint:* multiply by $\sin \frac{\pi}{7}$ and use $2 \sin u \cos u = \sin 2u$ three times.

7.37 Evaluate $\tan 20^\circ \tan 40^\circ \tan 80^\circ$ exactly, using $\tan \theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$.

7.38 Evaluate $\sin^2 1^\circ + \sin^2 2^\circ + \cdots + \sin^2 89^\circ$.

7.39 Solve $\sin^4 x + \cos^4 x = \frac{5}{8}$ on $[0, 2\pi)$.

7.40 Prove that $\sin x + \sin 2x + \cdots + \sin nx = \frac{\sin \frac{nx}{2} \sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}}$ for $\sin \frac{x}{2} \neq 0$.

CHAPTER 8

SOLVING TRIANGLES

LEARNING OBJECTIVES

- Prove and apply the Law of Sines and the Law of Cosines.
- Analyze the ambiguous case (SSA), determining whether zero, one, or two triangles exist.
- Compute the area of a triangle from two sides and the included angle, and from three sides (Heron's formula).
- Solve applied problems in navigation, surveying, and physics.

Right-triangle trigonometry solves right triangles. Most triangles that arise in practice, from a surveyor's measurements to the forces on a bridge, are not right triangles. Two laws extend trigonometry to every triangle. Throughout, a triangle has angles A, B, C and opposite sides a, b, c .

8.1 THE LAW OF SINES

Theorem 8.1 (Law of Sines). *In any triangle,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Proof. Drop the altitude h from C to the line AB . In the right triangles it creates, $h = b \sin A$ and $h = a \sin B$ (if B is obtuse, the foot of the altitude lies outside the triangle and $h = a \sin(\pi - B) = a \sin B$). So $b \sin A = a \sin B$, which is the first equality. The altitude from A gives the second. ■

The Law of Sines solves a triangle when we know two angles and a side (AAS or ASA), and sometimes when we know two sides and an angle opposite one of them (SSA).

Example 8.2 (AAS). Solve the triangle with $A = 40^\circ$, $B = 60^\circ$, $a = 10$.

Solution. $C = 180^\circ - 100^\circ = 80^\circ$. Then $b = \frac{a \sin B}{\sin A} = \frac{10 \sin 60^\circ}{\sin 40^\circ} \approx 13.47$ and $c = \frac{10 \sin 80^\circ}{\sin 40^\circ} \approx 15.32$.

8.1.1 THE AMBIGUOUS CASE

Given a , b , and the angle A opposite a , imagine side b fixed with angle A at its left end, and side a swinging from the other end like a pendulum. Let $h = b \sin A$ be the distance from that end to the ray where side c must lie.

Theorem 8.3 (SSA cases, with A acute). If $a < h$, no triangle exists. If $a = h$, exactly one (right) triangle exists. If $h < a < b$, two triangles exist. If $a \geq b$, exactly one triangle exists. When A is obtuse or right, there is one triangle if $a > b$ and none otherwise.

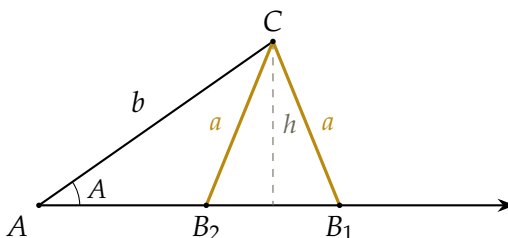


Figure 8.1: The ambiguous case $h < a < b$: side a reaches the base at two points, giving two triangles AB_1C and AB_2C .

Example 8.4 (Two triangles). Solve the triangle with $A = 43^\circ$, $a = 7$, $b = 10$.

Solution. $h = 10 \sin 43^\circ \approx 6.82 < 7 < 10$: two triangles. $\sin B = \frac{10 \sin 43^\circ}{7} \approx 0.9743$, so $B_1 \approx 76.98^\circ$ or $B_2 = 180^\circ - B_1 \approx 103.02^\circ$. First triangle: $C_1 \approx 60.02^\circ$, $c_1 = \frac{7 \sin C_1}{\sin 43^\circ} \approx 8.89$. Second: $C_2 \approx 33.98^\circ$, $c_2 \approx 5.74$.

COMMON ERROR (The calculator returns one angle). $\arcsin$ returns an angle in $[-90^\circ, 90^\circ]$, so it never gives the obtuse solution $180^\circ - B$. In every SSA problem, check whether the supplement also fits: it does exactly when $A + (180^\circ - B) < 180^\circ$.

8.2 THE LAW OF COSINES

Theorem 8.5 (Law of Cosines). *In any triangle,*

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

and similarly $a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = a^2 + c^2 - 2ac \cos B$.

Proof. Place C at the origin and B at $(a, 0)$. Then $A = (b \cos C, b \sin C)$, and by the distance formula $c^2 = (b \cos C - a)^2 + (b \sin C)^2 = b^2(\cos^2 C + \sin^2 C) - 2ab \cos C + a^2$. ■

When $C = 90^\circ$, $\cos C = 0$ and the law reduces to the Pythagorean Theorem; the term $-2ab \cos C$ is the correction for a non-right angle. The Law of Cosines solves triangles given two sides and the included angle (SAS) or three sides (SSS), and it has no ambiguous case, because $\arccos$ returns angles in $[0^\circ, 180^\circ]$, the full range of possible triangle angles.

Example 8.6 (SAS). Two ships leave port at the same time. One sails at 20 mi/h on a bearing of N30°E, the other at 15 mi/h on a bearing of S80°E. How far apart are they after 2 hours?

Solution. After 2 hours the sides are 40 and 30 miles. The angle between the courses is $180^\circ - 30^\circ - 80^\circ = 70^\circ$. So $d^2 = 40^2 + 30^2 - 2(40)(30) \cos 70^\circ \approx 1679.2$, and $d \approx 40.98$ miles.

Example 8.7 (SSS). Find the largest angle of the triangle with sides 5, 7, 9.

Solution. The largest angle is opposite the longest side: $\cos C = \frac{25+49-81}{2 \cdot 5 \cdot 7} = -\frac{7}{70} = -\frac{1}{10}$, so $C = \arccos\left(-\frac{1}{10}\right) \approx 95.74^\circ$.

8.3 AREA OF A TRIANGLE

Theorem 8.8 (Area formulas). *The area of a triangle is $K = \frac{1}{2}ab \sin C$. In terms of the three sides,*

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \frac{a+b+c}{2} \quad (\text{Heron's formula}).$$

Proof. With base a , the height is $b \sin C$, giving the first formula. For Heron's, square it and use the Law of Cosines:

$$\begin{aligned} 16K^2 &= 4a^2b^2(1 - \cos^2 C) = 4a^2b^2 - (a^2 + b^2 - c^2)^2 \\ &= (2ab - a^2 - b^2 + c^2)(2ab + a^2 + b^2 - c^2). \end{aligned}$$

The factors are $(c - a + b)(c + a - b)$ and $(a + b - c)(a + b + c)$, which equal $2(s - a) \cdot 2(s - b) \cdot 2(s - c) \cdot 2s$. Dividing by 16 gives the formula. ■

Example 8.9 (Heron's formula). For sides 5, 7, 9: $s = 10.5$ and $K = \sqrt{10.5 \cdot 5.5 \cdot 3.5 \cdot 1.5} = \sqrt{303.1875} \approx 17.41$. Check with the angle found in Example 8.7: $\frac{1}{2} \cdot 5 \cdot 7 \sin 95.74^\circ \approx 17.41$.

8.4 APPLICATIONS

Example 8.10 (Surveying across a river). To find the distance from point A on one bank to a tree T on the other, a surveyor measures a baseline $AB = 120$ m along the bank and the angles $\angle TAB = 67^\circ$ and $\angle TBA = 51^\circ$. Find AT .

Solution. $\angle ATB = 180^\circ - 67^\circ - 51^\circ = 62^\circ$. By the Law of Sines, $AT = \frac{120 \sin 51^\circ}{\sin 62^\circ} \approx 105.62$ m.

Example 8.11 (Resultant force). Two forces of 30 N and 50 N act on an object, with an angle of 40° between their directions. Find the magnitude of the resultant.

Solution. By the parallelogram rule, the resultant is the diagonal of the parallelogram with sides 30 and 50; in the triangle it forms, the angle between the sides is $180^\circ - 40^\circ = 140^\circ$. So $R^2 = 30^2 + 50^2 - 2(30)(50) \cos 140^\circ \approx 5698.1$ and $R \approx 75.49$ N. Vectors (Chapter 10) make such computations routine.

LOOKING AHEAD. The Law of Cosines is the dot product in disguise: for vectors $\mathbf{u}$ and $\mathbf{v}$, $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}$, so $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$. Volume IV opens with this identity, and it reappears in quantum mechanics as the inner product $\langle \phi | \psi \rangle$, whose square gives measurement probabilities.

8.5 WORKED PROBLEMS

These problems apply the Laws of Sines and Cosines in combination. The main decision in each is which law to use first.

PROBLEM 8.1 How many triangles?

Given $A = 35^\circ$ and $b = 20$, for which lengths a are there no triangles, one triangle, or two triangles?

Solution. The height from the far end of b to the other side is $h = 20 \sin 35^\circ \approx 11.47$. By Theorem 8.3: no triangle if $a < 11.47$; exactly one (a right triangle) if $a = 20 \sin 35^\circ$; two triangles if $11.47 < a < 20$; and exactly one if $a \geq 20$, because then the swinging side reaches the base only once on the correct side.

PROBLEM 8.2 A course with a turn

A ship sails 30 nautical miles on a bearing of 060° , then turns and sails 45 nautical miles on a bearing of 150° . How far is it from its starting point, and what bearing should it steer to return?

Solution. The two courses differ by 90° , so the turn makes a right angle and the distance is $\sqrt{30^2 + 45^2} \approx 54.08$ nmi. (In general, the angle inside the triangle at the turn is 180° minus the change in course, and the Law of Cosines is needed.) In coordinates with north as the y -axis, the ship ends at $(30 \sin 60^\circ + 45 \sin 150^\circ, 30 \cos 60^\circ + 45 \cos 150^\circ) \approx (48.48, -23.97)$. Its bearing from the start is about 116.3° , so it must steer $116.3^\circ + 180^\circ = 296.3^\circ$ to return.

PROBLEM 8.3 Area and inscribed circle

Find the area of the triangle with sides 7, 8, 9, and the radius of its inscribed circle.

Solution. $s = 12$ and Heron's formula gives $K = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = \sqrt{720} = 12\sqrt{5} \approx 26.83$. The inscribed circle touches all three sides; joining its center to the vertices splits the triangle into three triangles of height r and bases a, b, c , so $K = \frac{1}{2}r(a + b + c) = rs$. Hence $r = \frac{K}{s} = \sqrt{5} \approx 2.24$.

PROBLEM 8.4 Height from two angles

From a point A on level ground the angle of elevation to the top of a tower is 40° . From a point B , 50 m closer, it is 60° . Find the height of the tower.

Solution. Let T be the top. In triangle ABT , the angle at B is $180^\circ - 60^\circ = 120^\circ$, so the angle at T is $180^\circ - 40^\circ - 120^\circ = 20^\circ$. By the Law of Sines, $BT = \frac{50 \sin 40^\circ}{\sin 20^\circ}$. In the right triangle at B , the height is $BT \sin 60^\circ = \frac{50 \sin 40^\circ \sin 60^\circ}{\sin 20^\circ} \approx 81.38$ m.

PROBLEM 8.5 The angle between two forces

Forces of 10 N and 14 N act on an object, and their resultant has magnitude 20 N. Find the angle between the forces.

Solution. In the force parallelogram, the resultant is a diagonal, and the triangle it forms with the two forces has sides 10, 14, 20, with the angle opposite 20 equal to $180^\circ - \theta$, where θ is the angle between the forces. By the Law of Cosines, $20^2 = 10^2 + 14^2 - 2(10)(14) \cos(180^\circ - \theta) = 296 + 280 \cos \theta$. So $\cos \theta = \frac{104}{280}$ and $\theta \approx 68.2^\circ$.

PROBLEM 8.6 A quadrilateral

In quadrilateral $ABCD$, $AB = 5$, $BC = 8$, $\angle B = 60^\circ$, $CD = 6$, and $DA = 4$. Find its area.

Solution. The diagonal AC splits it into two triangles. In triangle ABC , $AC^2 = 25 + 64 - 2(5)(8) \cos 60^\circ = 49$, so $AC = 7$, and its area is $\frac{1}{2}(5)(8) \sin 60^\circ = 10\sqrt{3}$. Triangle ACD has sides 7, 6, 4, so $s = 8.5$ and its area is $\sqrt{8.5 \cdot 1.5 \cdot 2.5 \cdot 4.5} \approx 11.98$. The total is $10\sqrt{3} + 11.98 \approx 29.30$.

PROBLEM 8.7 The circumscribed circle

A triangle has $a = 10$ and $A = 30^\circ$. Find the radius of its circumscribed circle, and explain why it does not depend on the other sides.

Solution. By the extended Law of Sines (Exercise 8.21), $\frac{a}{\sin A} = 2R$, so $R = \frac{10}{2 \sin 30^\circ} = 10$. Every triangle with a side 10 opposite a 30° angle is inscribed in a circle of radius 10: an inscribed angle is half the central angle on the same chord, so a 30° angle sees a chord of length 10 only on a circle of radius 10.

SKILL DRILLS

Short problems for building speed and accuracy. Round to two decimals. Answers follow the list.

- | | |
|--|---|
| D8.1 $A = 30^\circ, B = 45^\circ, a = 10$. Find b . | D8.12 Area with $a = 6, b = 9, C = 30^\circ$. |
| D8.2 $A = 50^\circ, C = 70^\circ, c = 12$. Find a . | D8.13 Area with sides 5, 5, 6. |
| D8.3 $A = 40^\circ, B = 75^\circ$. Find C . | D8.14 Area with sides 9, 10, 17. |
| D8.4 $a = 5, b = 7, C = 60^\circ$. Find c . | D8.15 $\sin B$ if $a = 4, b = 6, A = 30^\circ$. |
| D8.5 $a = 8, b = 8, C = 90^\circ$. Find c . | D8.16 $\cos A$ if $a = 7, b = 5, c = 8$. |
| D8.6 Sides 3, 5, 7. Find the largest angle. | D8.17 Perimeter of an isosceles triangle with legs 10 and apex angle 40° . |
| D8.7 Sides 6, 6, 6. Find each angle. | D8.18 Diagonal of a rhombus with side 6 and angle 120° (the longer diagonal). |
| D8.8 $A = 30^\circ, a = 4, b = 8$. How many triangles? | D8.19 $2R$ for a triangle with $a = 9, A = 60^\circ$. |
| D8.9 $A = 30^\circ, a = 3, b = 8$. How many triangles? | D8.20 Third side when two sides are 3 and 4 and the included angle is 90° . |
| D8.10 $A = 30^\circ, a = 10, b = 8$. How many triangles? | |
| D8.11 $A = 120^\circ, a = 5, b = 7$. How many triangles? | |

ANSWERS TO SKILL DRILLS

D8.1 14.14. D8.2 9.78. D8.3 65° . D8.4 6.24. D8.5 11.31. D8.6 120° . D8.7 60° each. D8.8 One (a right triangle). D8.9 None. D8.10 One. D8.11 None. D8.12 13.50. D8.13 12. D8.14 36. D8.15 0.75. D8.16 0.5, so $A = 60^\circ$. D8.17 26.84. D8.18 10.39. D8.19 10.39. D8.20 5.

EXERCISES

COMPUTATION

- 8.1** Solve the triangle with $A = 50^\circ, B = 68^\circ, c = 230$.

- 8.2** Solve the triangle with $B = 29^\circ$, $C = 51^\circ$, $b = 44$.
- 8.3** How many triangles have $A = 30^\circ$, $a = 5$, $b = 12$? Explain.
- 8.4** How many triangles have $A = 30^\circ$, $a = 6$, $b = 12$? Solve it.
- 8.5** Solve all triangles with $A = 36^\circ$, $a = 8$, $b = 12$.
- 8.6** Solve all triangles with $B = 115^\circ$, $b = 20$, $c = 14$.
- 8.7** Solve all triangles with $C = 40^\circ$, $c = 10$, $a = 25$.
- 8.8** Solve the triangle with $a = 10$, $b = 12$, $C = 40^\circ$.
- 8.9** Solve the triangle with $a = 4$, $b = 5$, $c = 7$.
- 8.10** Find the angles of the triangle with sides 10, 10, 16.
- 8.11** Find the area of the triangle with $a = 12$, $b = 15$, $C = 110^\circ$.
- 8.12** Find the area of the triangle with sides 13, 14, 15.
- 8.13** A parallelogram has sides 8 and 12 and an angle of 60° . Find both diagonals and the area.
- 8.14** A pilot flies 200 mi on a bearing of $N20^\circ E$, then 150 mi on a bearing of $S70^\circ E$. Find the distance from the starting point, and the bearing of the final position from the starting point.
- 8.15** Two observers 5 km apart on a straight road see a balloon between them at angles of elevation 32° and 48° . Find the balloon's altitude.
- 8.16** A 40-ft telephone pole on a hillside inclined 12° to the horizontal casts a shadow straight down the hill. The angle of elevation of the sun is 50° . Find the length of the shadow.
- 8.17** A triangular lot has sides 120 m, 150 m, and 200 m. Find its area and its largest angle.
- 8.18** Find the length of the side of a regular pentagon inscribed in a circle of radius 10.
- 8.19** Two forces of 12 N and 18 N have a resultant of 25 N. Find the angle between the forces.
- 8.20** A tunnel is to be built through a mountain between A and B . From a point C , $CA = 388$ m, $CB = 212$ m, and $\angle ACB = 82.4^\circ$. Find the tunnel's length.

CONCEPTS AND PROOF

- 8.21** Prove that $\frac{a}{\sin A} = 2R$, where R is the radius of the circumscribed circle.
Hint: inscribe the triangle and use the diameter through a vertex.
- 8.22** Show that the Law of Cosines implies the Pythagorean Theorem and its converse.
- 8.23** Prove that the area of a triangle is $K = \frac{abc}{4R}$.
- 8.24** Show that $a = b \cos C + c \cos B$ (the projection formula).
- 8.25** Explain why SSA can be ambiguous but SAS cannot.
- 8.26** Prove the Law of Tangents: $\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$.
- 8.27** Show that in any triangle, $\cos A + \cos B + \cos C \leq \frac{3}{2}$, with equality for the equilateral triangle. (You may assume the result that the maximum occurs when $A = B = C$.)
- 8.28** Derive the formula $K = \frac{a^2 \sin B \sin C}{2 \sin A}$.
- 8.29** For which triangles does Heron's formula give $K = 0$? Interpret.
- 8.30** Prove that the angle bisector from C divides c in the ratio $a : b$, using the Law of Sines.

CHALLENGE

- 8.31** Find the radius of the inscribed circle of the 13–14–15 triangle, using $K = rs$.
- 8.32** A triangle has sides $a = 7$, $b = 8$, and the angle opposite a is $A = 60^\circ$. Find c (there are two answers).
- 8.33** Three circles of radii 1, 2, 3 are mutually externally tangent. Find the area of the triangle formed by their centers, and the angles of that triangle.
- 8.34** In triangle ABC , $a = 5$, $b = 7$, and $\cos(A - B) = \frac{31}{32}$. Find c .
- 8.35** Prove that in any triangle, $\frac{a^2 + b^2 + c^2}{2abc} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$.



PART III

ANALYTIC GEOMETRY AND ALGEBRA

Complex numbers, polar coordinates and vectors, conic sections, systems and matrices, and sequences and series.

CHAPTER 9

COMPLEX NUMBERS

LEARNING OBJECTIVES

- Add, subtract, multiply, and divide complex numbers, and use conjugates.
- Plot complex numbers and compute modulus and argument.
- Write complex numbers in trigonometric form, and multiply and divide them geometrically.
- Prove and apply De Moivre's Theorem to compute powers.
- Find all n th roots of a complex number, including the roots of unity.

The equation $x^2 = -1$ has no real solution. Rather than accept that, mathematicians in the sixteenth century began to calculate with a symbol i satisfying $i^2 = -1$, at first reluctantly, because it produced correct real answers to cubic equations. The resulting number system, the complex numbers, turned out to be the natural home of algebra (every polynomial factors completely, Theorem 3.11) and of much of physics: quantum mechanics is written in complex numbers from its first equation.

9.1 ARITHMETIC OF COMPLEX NUMBERS

Definition 9.1 (Complex numbers). A *complex number* is an expression $z = a + bi$ with a, b real and $i^2 = -1$. $a = \operatorname{Re} z$ is the *real part* and $b = \operatorname{Im} z$ the *imaginary part*. Two complex numbers are equal when their real parts and their imaginary parts are equal. The *conjugate* of z is $\bar{z} = a - bi$.

Complex numbers are added and multiplied like polynomials in i , replacing i^2 by -1 :

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The key fact about conjugates is $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$, a nonnegative *real* number. Division uses it: multiply numerator and denominator by the conjugate of the denominator.

Example 9.2 (Arithmetic). (a) $(3 - 2i)(4 + 5i) = 12 + 15i - 8i - 10i^2 = 22 + 7i$. (b) $\frac{2 + 3i}{1 - 2i} = \frac{(2 + 3i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{2 + 4i + 3i + 6i^2}{5} = \frac{-4 + 7i}{5} = -\frac{4}{5} + \frac{7}{5}i$. (c) Powers of i repeat with period 4: $i, -1, -i, 1$. So $i^{2026} = i^{4 \cdot 506 + 2} = i^2 = -1$.

Theorem 9.3 (Properties of conjugates). $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $\bar{\bar{z}} = z$, and z is real if and only if $z = \bar{z}$.

These properties prove the Conjugate Zeros Theorem (Theorem 3.12): if P has real coefficients and $P(z) = 0$, then $P(\bar{z}) = \overline{P(z)} = \bar{0} = 0$.

COMMON ERROR ($\sqrt{-a}\sqrt{-b} \neq \sqrt{ab}$). For $a, b > 0$, write $\sqrt{-a} = i\sqrt{a}$ before multiplying. $\sqrt{-4}\sqrt{-9} = (2i)(3i) = 6i^2 = -6$, not $\sqrt{36} = 6$. The rule $\sqrt{x}\sqrt{y} = \sqrt{xy}$ holds only for $x, y \geq 0$.

9.2 THE COMPLEX PLANE

The complex number $a + bi$ corresponds to the point (a, b) in the plane, called the *complex plane*: the horizontal axis is the *real axis* and the vertical axis the *imaginary axis*. Addition is vector addition (the parallelogram rule), and conjugation is reflection in the real axis.

Definition 9.4 (Modulus). The *modulus* (or absolute value) of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$, its distance from the origin. Note $|z|^2 = z\bar{z}$.

Theorem 9.5 (Properties of modulus). $|zw| = |z||w|$, $|z/w| = |z|/|w|$, and $|z + w| \leq |z| + |w|$ (the triangle inequality). $|z - w|$ is the distance between z and w .

Proof of the first. $|zw|^2 = zw\bar{zw} = z\bar{z}w\bar{w} = |z|^2|w|^2$. ■

9.3 TRIGONOMETRIC FORM

A point at distance r from the origin in the direction of angle θ is $(r \cos \theta, r \sin \theta)$. So every nonzero complex number can be written

$$z = r(\cos \theta + i \sin \theta), \quad r = |z|, \quad \tan \theta = \frac{b}{a}.$$

θ is an *argument* of z , determined up to multiples of 2π ; choose it in the quadrant where (a, b) lies.

Example 9.6 (Converting). $1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$; $-\sqrt{3} + i = 2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$; $-3i = 3(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})$.

Theorem 9.7 (Multiplication and division). If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)], \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad (z_2 \neq 0). \end{aligned}$$

Proof. Multiply out: the real part is $r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) = r_1 r_2 \cos(\theta_1 + \theta_2)$ and the imaginary part is $r_1 r_2 (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) = r_1 r_2 \sin(\theta_1 + \theta_2)$, by the addition formulas. Division follows because $\frac{z_1}{z_2} \cdot z_2 = z_1$. ■

To multiply complex numbers, multiply the moduli and add the arguments. Multiplication by $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ is a rotation by 90° ; multiplication by any z is a rotation by $\arg z$ combined with a scaling by $|z|$.

9.4 DE MOIVRE'S THEOREM

Theorem 9.8 (De Moivre's Theorem). For every positive integer n , $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$.

Proof. Induction on n , using Theorem 9.7: if it holds for n , then multiplying by $r(\cos \theta + i \sin \theta)$ multiplies the modulus by r and adds θ to the argument. ■

Example 9.9 (A large power). Compute $(1 + i)^{10}$ and $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^9$.

Solution. $(1 + i)^{10} = (\sqrt{2})^{10} (\cos \frac{10\pi}{4} + i \sin \frac{10\pi}{4}) = 32 (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 32i$. The second number is $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$, so its ninth power is $\cos 3\pi + i \sin 3\pi = -1$.

Example 9.10 (Identities from De Moivre). Expanding $(\cos \theta + i \sin \theta)^3$ by the binomial theorem and comparing with $\cos 3\theta + i \sin 3\theta$: $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = 4 \cos^3 \theta - 3 \cos \theta$ and $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$. Multiple-angle formulas come for free.

9.5 ROOTS OF COMPLEX NUMBERS

Theorem 9.11 (*n*th roots). A nonzero complex number $z = r(\cos \theta + i \sin \theta)$ has exactly *n* distinct *n*th roots:

$$w_k = r^{1/n} \left[\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right], \quad k = 0, 1, \dots, n-1.$$

Proof. By De Moivre, $w_k^n = r(\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)) = z$. The arguments $\frac{\theta + 2k\pi}{n}$ for $k = 0, \dots, n-1$ differ by less than 2π , so the w_k are distinct; and any *n*th root $w = \rho(\cos \phi + i \sin \phi)$ must have $\rho^n = r$ and $n\phi = \theta + 2k\pi$ for some integer k , which gives one of these. ■

The *n* roots lie on the circle of radius $r^{1/n}$, equally spaced at angles $\frac{2\pi}{n}$: they are the vertices of a regular *n*-gon.

Example 9.12 (Cube roots). Find the cube roots of -8 .

Solution. $-8 = 8(\cos \pi + i \sin \pi)$. The roots are $2 \left[\cos \frac{\pi + 2k\pi}{3} + i \sin \frac{\pi + 2k\pi}{3} \right]$ for $k = 0, 1, 2$: $2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = 1 + \sqrt{3}i$, $2(\cos \pi + i \sin \pi) = -2$, and $2 \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) = 1 - \sqrt{3}i$.

Example 9.13 (Roots of unity). The *n*th roots of 1 are ω^k , $k = 0, \dots, n-1$, where $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. For $n = 4$ they are $1, i, -1, -i$. Their sum is 0

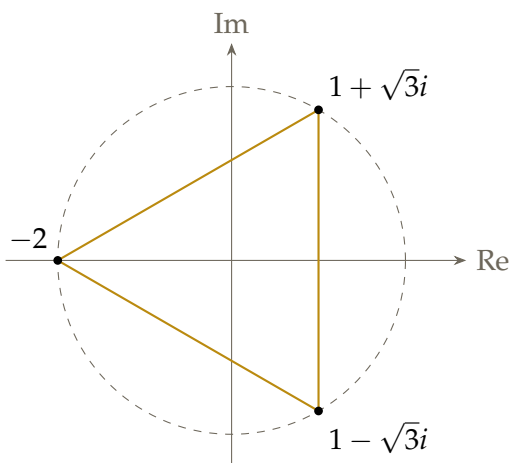


Figure 9.1: The three cube roots of -8 form an equilateral triangle on the circle of radius 2.

for every $n \geq 2$: $\sum_{k=0}^{n-1} \omega^k = \frac{\omega^n - 1}{\omega - 1} = 0$.

Example 9.14 (A quadratic with complex coefficients). Solve $z^2 - (3 + i)z + (2 + 2i) = 0$.

Solution. The quadratic formula works for complex coefficients. The discriminant is $(3 + i)^2 - 4(2 + 2i) = 8 + 6i - 8 - 8i = -2i$. Its square roots are $\pm(1 - i)$, since $(1 - i)^2 = -2i$. So $z = \frac{(3+i) \pm (1-i)}{2}$: $z = 2$ or $z = 1 + i$.

LOOKING AHEAD. Volume III proves Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ from power series, so the trigonometric form becomes $z = re^{i\theta}$, and Theorem 9.7 becomes the exponent law $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$. In Volume IV, every quantum state is a vector of complex numbers, the time evolution of a state of energy E is multiplication by $e^{-iEt/\hbar}$ (a rotation in the complex plane), and the probability of an outcome is a modulus squared, $|c|^2 = c\bar{c}$.

9.6 WORKED PROBLEMS

These problems use the algebra and the geometry of complex numbers together. Converting to trigonometric form is usually the key step.

PROBLEM 9.1 A quadratic with complex coefficients

Solve $z^2 + 2iz - 5 = 0$.

Solution. The quadratic formula holds for complex coefficients. The discriminant is $(2i)^2 - 4(1)(-5) = -4 + 20 = 16$, so $z = \frac{-2i \pm 4}{2} = \pm 2 - i$. Check $z = 2 - i$: $(2 - i)^2 + 2i(2 - i) - 5 = (3 - 4i) + (4i + 2) - 5 = 0$. Unlike real quadratics, the two roots here are not conjugates, because the coefficients are not real.

PROBLEM 9.2 A quotient of powers

Compute $\frac{(1 + \sqrt{3}i)^8}{(1 - i)^6}$ in the form $a + bi$.

Solution. Expanding would be slow; trigonometric form is fast. $1 + \sqrt{3}i = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$ and $1 - i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$. By De Moivre, the numerator has modulus $2^8 = 256$ and argument $\frac{8\pi}{3}$, and the denominator has modulus $(\sqrt{2})^6 = 8$ and argument $-\frac{3\pi}{2}$. Dividing: modulus 32, argument $\frac{8\pi}{3} + \frac{3\pi}{2} = \frac{25\pi}{6}$, which is coterminal with $\frac{\pi}{6}$. So the result is $32(\frac{\sqrt{3}}{2} + \frac{1}{2}i) = 16\sqrt{3} + 16i$.

PROBLEM 9.3 Cube roots in rectangular form

Solve $z^3 = -27i$.

Solution. $-27i = 27(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})$. The cube roots have modulus 3 and arguments $\frac{3\pi/2 + 2k\pi}{3} = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$:

$$z = 3i, \quad z = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i, \quad z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

Check the first: $(3i)^3 = 27i^3 = -27i$. The three roots form an equilateral triangle on the circle of radius 3.

PROBLEM 9.4 Factoring with complex roots

Factor $x^4 + 4$ into real quadratic factors.

Solution. The zeros are the fourth roots of $-4 = 4(\cos \pi + i \sin \pi)$: modulus $\sqrt{2}$, arguments $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, that is $\pm 1 \pm i$. Pair each with its conjugate: $(x - (1 + i))(x - (1 - i)) = x^2 - 2x + 2$ and $(x - (-1 + i))(x - (-1 - i)) = x^2 + 2x + 2$. So $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$, which can be checked

by multiplying out.

PROBLEM 9.5 Rotation by multiplication

Rotate the point $(3, 1)$ by 60° counterclockwise about the origin.

Solution. Multiplying by $\cos 60^\circ + i \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ rotates by 60° : $(3 + i) \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i + \frac{1}{2}i - \frac{\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2} + \frac{3\sqrt{3}+1}{2}i$. The rotated point is $\left(\frac{3-\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2} \right) \approx (0.634, 3.098)$. Its distance from the origin is still $\sqrt{10}$.

PROBLEM 9.6 Equations involving modulus

Find all z with $|z| = |z - 2|$ and $|z - i| = 1$.

Solution. $|z| = |z - 2|$ says z is equidistant from 0 and 2, so it lies on the perpendicular bisector $\operatorname{Re} z = 1$. Write $z = 1 + yi$. Then $|z - i| = |1 + (y - 1)i| = \sqrt{1 + (y - 1)^2} = 1$ forces $y = 1$. The only solution is $z = 1 + i$: the line is tangent to the circle.

PROBLEM 9.7 A multiple-angle formula

Use De Moivre's Theorem to express $\cos 4\theta$ in terms of $\cos \theta$.

Solution. Write $c = \cos \theta$, $s = \sin \theta$. The real part of $(c + is)^4$ is $c^4 - 6c^2s^2 + s^4$ by the binomial theorem (the terms with i^0, i^2, i^4). Replace s^2 by $1 - c^2$:

$$\cos 4\theta = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2 = 8c^4 - 8c^2 + 1.$$

Check at $\theta = 0$: $8 - 8 + 1 = 1 = \cos 0$.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D9.1 $(2 + 5i) + (3 - 7i)$.

D9.6 $\overline{6 - 5i}$.

D9.2 $(4 - i) - (1 - 3i)$.

D9.7 $(3 + 4i)(3 - 4i)$.

D9.3 $(2 + i)(3 - i)$.

D9.8 $\frac{5}{2 - i}$.

D9.4 $(1 + 2i)^2$.

D9.9 $\frac{1 + 3i}{1 + i}$.

D9.5 i^{11} .

- D9.10** $|-5 + 12i|$. **D9.17** $(\text{cis } 15^\circ)^6$ in rectangular form.
- D9.11** $|2i|$. **D9.18** $(2i)^5$.
- D9.12** Argument of $1 + i$. **D9.19** The two square roots of -9 .
- D9.13** Argument of $-\sqrt{3} - i$. **D9.20** The two square roots of $2i$.
- D9.14** Trigonometric form of -2 . **D9.21** How many fifth roots does 1 have?
- D9.15** $2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$ in rectangular form. **D9.22** Solve $z^2 + 9 = 0$.
- D9.16** Modulus and argument of $z_1 z_2$ if $z_1 = 3 \text{ cis } 20^\circ$, $z_2 = 2 \text{ cis } 40^\circ$ (where $\text{cis } \theta = \cos \theta + i \sin \theta$). **D9.23** Solve $z^2 - 2z + 2 = 0$.
- D9.24** Sum of the cube roots of 8.

ANSWERS TO SKILL DRILLS

D9.1 $5 - 2i$. D9.2 $3 + 2i$. D9.3 $7 + i$. D9.4 $-3 + 4i$. D9.5 $-i$. D9.6 $6 + 5i$.
D9.7 25. D9.8 $2 + i$. D9.9 $2 + i$. D9.10 13. D9.11 2. D9.12 $\frac{\pi}{4}$. D9.13 $\frac{7\pi}{6}$.
D9.14 $2(\cos \pi + i \sin \pi)$. D9.15 $\sqrt{3} + i$. D9.16 modulus 6, argument 60° .
D9.17 i . D9.18 $32i$. D9.19 $\pm 3i$. D9.20 $\pm(1 + i)$. D9.21 5. D9.22 $\pm 3i$. D9.23
 $1 \pm i$. D9.24 0.

EXERCISES

COMPUTATION

- 9.1** Simplify $(5 - 3i) - (2 + 7i)$, $(4 + i)(2 - 3i)$, and $(1 + i)^2(1 - i)$.
- 9.2** Write in the form $a + bi$: $\frac{3 - 4i}{2 + i}$, $\frac{1}{i}$, $\frac{1 + i}{1 - i}$.
- 9.3** Compute i^{57} , i^{-3} , and $1 + i + i^2 + \cdots + i^{100}$.
- 9.4** Solve $x^2 + 4x + 13 = 0$ and $2x^2 - 2x + 5 = 0$.
- 9.5** Find the modulus of $3 - 4i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $(2 + i)^4$.
- 9.6** Plot $z = 2 + 3i$, $\bar{z}$, $-z$, and iz , and describe the geometric effect of multiplying by i .
- 9.7** Describe the set of z with $|z - 2i| = 3$ and the set with $|z - 1| = |z + 1|$.
- 9.8** Write in trigonometric form: $-1 - i$, $2\sqrt{3} - 2i$, -5 , $4i$.
- 9.9** Write in the form $a + bi$: $6\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$ and $\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$.

- 9.10** Compute $z_1 z_2$ and $\frac{z_1}{z_2}$ in trigonometric form for $z_1 = 4 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$, $z_2 = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$.
- 9.11** Compute $(1 - i)^8$ using De Moivre's Theorem.
- 9.12** Compute $\left(\sqrt{3} + i \right)^6$ and $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)^{15}$.
- 9.13** Compute $(2 - 2i)^{-4} = \frac{1}{(2 - 2i)^4}$ in the form $a + bi$.
- 9.14** Find the square roots of i and of -4 .
- 9.15** Find the cube roots of $27i$.
- 9.16** Find the fourth roots of -16 .
- 9.17** Find all fourth roots of $-8 - 8\sqrt{3}i$ in rectangular form.
- 9.18** Find the fifth roots of unity, and show that they sum to 0.
- 9.19** Solve $z^3 = 1 + i$, giving the answers in trigonometric form.
- 9.20** Solve $z^6 + 64 = 0$.

CONCEPTS AND PROOF

- 9.21** Prove that $\overline{z\bar{w}} = \bar{z}\bar{w}$ and $\overline{z + w} = \bar{z} + \bar{w}$.
- 9.22** Prove that $\operatorname{Re} z = \frac{z + \bar{z}}{2}$ and $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$.
- 9.23** Prove that $|z + w| \leq |z| + |w|$ by squaring and using $z\bar{w} + \bar{z}w = 2 \operatorname{Re}(z\bar{w}) \leq 2|z||w|$.
- 9.24** Use De Moivre's Theorem to derive formulas for $\cos 4\theta$ and $\sin 4\theta$.
- 9.25** Show that the nonreal cube roots of unity, ω and ω^2 , satisfy $\omega^2 + \omega + 1 = 0$.
- 9.26** Prove that if $z^n = 1$ then $|z| = 1$.
- 9.27** Show that the n th roots of any nonzero z are $w_0, w_0\omega, w_0\omega^2, \dots, w_0\omega^{n-1}$, where ω is a primitive n th root of unity.
- 9.28** Explain why the complex numbers cannot be ordered compatibly with multiplication, as the reals are. *Hint:* is $i > 0$ or $i < 0$?
- 9.29** Show that multiplication by $\cos \theta + i \sin \theta$ rotates every point of the plane by θ about the origin.

9.30 Prove that a real polynomial of odd degree has at least one real zero, using the Conjugate Zeros Theorem.

CHALLENGE

9.31 Compute $(1+i)^{100}$, and use the binomial theorem to deduce $\binom{100}{0} - \binom{100}{2} + \binom{100}{4} - \cdots + \binom{100}{100}$.

9.32 Find all complex z with $z^2 = \bar{z}$.

9.33 Prove that $\cos \frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ using the fact that $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ satisfies $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$. *Hint:* divide by ω^2 and let $u = \omega + \omega^{-1}$.

9.34 Find the product of all the n th roots of unity.

9.35 Show that $\sum_{k=0}^{n-1} \cos \frac{2k\pi}{n} = 0$ for $n \geq 2$.

CHAPTER 10

POLAR COORDINATES, PARAMETRIC EQUATIONS, AND VECTORS

LEARNING OBJECTIVES

- Convert between polar and rectangular coordinates and equations, and sketch polar curves.
- Describe curves with parametric equations, eliminate the parameter, and model projectile motion.
- Represent vectors in component form, and add, scale, and normalize them.
- Compute dot products, angles between vectors, projections, and work.

Rectangular coordinates are not always the natural description. A radar screen locates objects by distance and direction; a planet's orbit is simplest as a relation between distance and angle; a moving object is best described by giving its position at each time. This chapter introduces three tools for these situations, and each one becomes central in calculus and physics.

10.1 POLAR COORDINATES

Definition 10.1 (Polar coordinates). The point P has polar coordinates (r, θ) if it lies at directed distance r from the origin (the *pole*) along the ray at angle θ from the positive x -axis. If $r < 0$, P lies at distance $|r|$ in the opposite direction. The conversion formulas are

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x} \quad (x \neq 0).$$

A point has infinitely many polar representations: (r, θ) , $(r, \theta + 2k\pi)$, and $(-r, \theta + \pi)$ all describe it. When converting (x, y) to polar form, choose θ in the quadrant of (x, y) .

Example 10.2 (Converting points and equations). (a) $(4, \frac{2\pi}{3})$ is $(-2, 2\sqrt{3})$. (b) $(-1, -1)$ is $(\sqrt{2}, \frac{5\pi}{4})$. (c) The equation $r = 6 \sin \theta$ becomes, after multiplying by r , $x^2 + y^2 = 6y$, that is $x^2 + (y - 3)^2 = 9$: a circle. (d) The line $x + y = 4$ becomes $r(\cos \theta + \sin \theta) = 4$, or $r = \frac{4}{\cos \theta + \sin \theta}$.

10.1.1 GRAPHS OF POLAR EQUATIONS

To sketch $r = f(\theta)$, tabulate r for θ in $[0, 2\pi]$ (watching where r is zero, maximal, or negative), and use symmetry: if replacing θ by $-\theta$ leaves the equation unchanged, the graph is symmetric about the x -axis.

Equation	Curve
$r = a$	circle of radius $ a $ centered at the pole
$\theta = \alpha$	line through the pole
$r = 2a \cos \theta, r = 2a \sin \theta$	circles through the pole
$r = a(1 \pm \cos \theta)$	cardioid
$r = a \pm b \cos \theta$	limaçon (inner loop if $ b > a $)
$r = a \cos n\theta$	rose: n petals (n odd) or $2n$ petals (n even)
$r^2 = a^2 \cos 2\theta$	lemniscate

COMMON ERROR (Missing intersections). Solving $f(\theta) = g(\theta)$ finds only intersections reached at the same θ on both curves. Two polar curves can meet at a point that one reaches at $\theta = \alpha$ and the other at a different angle, and both may pass through the pole at different θ . Always sketch both

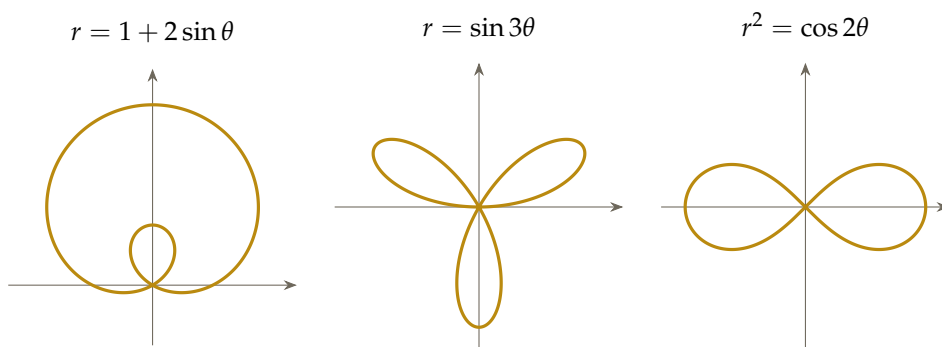


Figure 10.1: A limaçon with an inner loop, a three-petaled rose, and a lemniscate.

graphs.

10.2 PARAMETRIC EQUATIONS

Definition 10.3 (Parametric curve). Equations $x = f(t)$, $y = g(t)$ for t in an interval define a *parametric curve*: the path of the point $(f(t), g(t))$ as t varies. The variable t is the *parameter*; as it increases, the point moves along the curve in a definite direction.

Example 10.4 (Eliminating the parameter). (a) $x = 2 + t$, $y = 3 - 2t$: from $t = x - 2$, $y = 3 - 2(x - 2) = 7 - 2x$, a line. (b) $x = 3 \cos t$, $y = 2 \sin t$, $0 \leq t \leq 2\pi$: $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2 t + \sin^2 t = 1$, an ellipse traced once counter-clockwise. (c) $x = \sqrt{t}$, $y = t$: $y = x^2$ with $x \geq 0$, only the right half of the parabola.

Example 10.5 (Projectile motion). A ball is thrown at speed v_0 and angle α above the horizontal. Ignoring air resistance, its position after t seconds is

$$x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2.$$

Find the range and show that the path is a parabola.

Solution. The ball lands when $y = 0$, at $t = \frac{2v_0 \sin \alpha}{g}$, so the range is $x = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$, largest when $\alpha = 45^\circ$. Eliminating $t = \frac{x}{v_0 \cos \alpha}$ gives $y = (\tan \alpha)x - \frac{g}{2v_0^2 \cos^2 \alpha}x^2$, a downward parabola.

10.3 VECTORS IN THE PLANE

A *vector* is a quantity with magnitude and direction, represented by an arrow; arrows with the same length and direction represent the same vector. Displacement, velocity, and force are vectors; mass and temperature are scalars.

Definition 10.6 (Component form). The vector from $P(x_1, y_1)$ to $Q(x_2, y_2)$ is $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$. For $\mathbf{v} = \langle a, b \rangle$, the *magnitude* is $|\mathbf{v}| = \sqrt{a^2 + b^2}$. Vectors are added and multiplied by scalars componentwise: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$ and $k\langle a, b \rangle = \langle ka, kb \rangle$.

A *unit vector* has magnitude 1; the unit vector in the direction of $\mathbf{v} \neq \mathbf{0}$ is $\frac{\mathbf{v}}{|\mathbf{v}|}$. The standard unit vectors are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$, so $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$. A vector of length r at angle θ is $\langle r \cos \theta, r \sin \theta \rangle$.

Example 10.7 (Navigation). A plane heads due north at 300 mi/h while a 40-mi/h wind blows from the west, that is, toward the east. Find the plane's true speed and direction.

Solution. Velocity relative to the ground: $\langle 40, 300 \rangle$. Speed $\sqrt{40^2 + 300^2} \approx 302.65$ mi/h, direction $\arctan \frac{40}{300} \approx 7.59^\circ$ east of north.

Example 10.8 (Equilibrium). A 100-lb weight hangs from two cables making angles of 50° and 32° with the horizontal. Find the tension in each.

Solution. Let the tensions be T_1 (left, 50°) and T_2 (right, 32°). Horizontal balance: $T_1 \cos 50^\circ = T_2 \cos 32^\circ$. Vertical balance: $T_1 \sin 50^\circ + T_2 \sin 32^\circ = 100$. Solving, $T_1 \approx 85.64$ lb and $T_2 \approx 64.91$ lb.

10.4 THE DOT PRODUCT

Definition 10.9 (Dot product). $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2$.

Theorem 10.10 (Geometric meaning). If θ is the angle between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ ($0 \leq \theta \leq \pi$), then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$. In particular, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular exactly when $\mathbf{u} \cdot \mathbf{v} = 0$.

Proof. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} - \mathbf{v}$ form a triangle. By the Law of Cosines, $|\mathbf{u} - \mathbf{v}|^2 =$

$|\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$. In components, the left side is $(u_1 - v_1)^2 + (u_2 - v_2)^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2(u_1v_1 + u_2v_2)$. Comparing gives the result. ■

Definition 10.11 (Projection and work). The *projection* of $\mathbf{u}$ onto $\mathbf{v}$ is $\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$. The work done by a constant force $\mathbf{F}$ moving an object along displacement $\mathbf{D}$ is $W = \mathbf{F} \cdot \mathbf{D}$.

Example 10.12 (Angle and projection). For $\mathbf{u} = \langle 3, 4 \rangle$ and $\mathbf{v} = \langle 5, -2 \rangle$: $\mathbf{u} \cdot \mathbf{v} = 15 - 8 = 7$, so $\cos\theta = \frac{7}{5\sqrt{29}}$ and $\theta \approx 74.93^\circ$. $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{7}{29} \langle 5, -2 \rangle$.

Example 10.13 (Work). A sled is pulled 50 m along level ground by a rope at 30° above horizontal with tension 80 N. The work is $W = 80 \cdot 50 \cos 30^\circ = 2000\sqrt{3} \approx 3464$ J.

LOOKING AHEAD. Polar coordinates become cylindrical and spherical coordinates in Volume IV, the natural setting for the hydrogen atom. Parametric curves become vector functions $\mathbf{r}(t)$, whose derivative is the velocity vector and whose integral of speed is arc length. And the dot product is the first of the two vector products studied there; its generalization, the inner product, is the central operation of quantum mechanics.

10.5 WORKED PROBLEMS

These problems move between polar, parametric and vector descriptions of the plane. In most of them the work is choosing which description makes the question easy.

PROBLEM 10.1 Identifying a polar curve

Identify the curve $r = 4 \sin \theta - 2 \cos \theta$.

Solution. Multiply by r to create r^2 , $r \sin \theta$ and $r \cos \theta$: $r^2 = 4r \sin \theta - 2r \cos \theta$, so $x^2 + y^2 = 4y - 2x$. Completing squares, $(x + 1)^2 + (y - 2)^2 = 5$. The curve is the circle with center $(-1, 2)$ and radius $\sqrt{5}$. Multiplying by r added only the pole, which is already on this circle (at θ with $\tan \theta = \frac{1}{2}$).

PROBLEM 10.2 Where two polar curves meet

Find all intersection points of $r = 1 + \cos \theta$ and $r = 3 \cos \theta$.

Solution. Setting the expressions equal gives $1 + \cos \theta = 3 \cos \theta$, so $\cos \theta = \frac{1}{2}$, $\theta = \pm \frac{\pi}{3}$, and $r = \frac{3}{2}$: the points $(\frac{3}{2}, \pm \frac{\pi}{3})$. Equating r values misses the pole, because each curve can pass through it at a different θ : the cardioid has $r = 0$ at $\theta = \pi$ and the circle at $\theta = \frac{\pi}{2}$. So the pole is a third intersection point. A sketch is the reliable way to catch points like this.

PROBLEM 10.3 A projectile

A ball is thrown from a height of 1.5 m at 30 m/s, 40° above horizontal. Using $g = 9.8 \text{ m/s}^2$ and ignoring air resistance, find its maximum height and where it lands.

Solution. The velocity components are $30 \cos 40^\circ \approx 22.98$ and $30 \sin 40^\circ \approx 19.28$, so

$$x = (30 \cos 40^\circ)t, \quad y = 1.5 + (30 \sin 40^\circ)t - 4.9t^2.$$

The parabola $y(t)$ peaks at $t = \frac{30 \sin 40^\circ}{9.8} \approx 1.97 \text{ s}$, where $y \approx 20.47 \text{ m}$. It lands when $y = 0$; the positive root of the quadratic is $t \approx 4.01 \text{ s}$, at $x \approx 92.20 \text{ m}$.

PROBLEM 10.4 Flying against a crosswind

A plane with airspeed 480 km/h must travel due east. The wind blows toward the south at 50 km/h. What heading should the pilot take, and what is the ground speed?

Solution. Let the plane point at angle α north of east, so its air velocity is $\langle 480 \cos \alpha, 480 \sin \alpha \rangle$, and the wind is $\langle 0, -50 \rangle$. For the ground velocity to point due east, the north components must cancel: $480 \sin \alpha = 50$, so $\alpha \approx 5.98^\circ$, a bearing of about 084.0° . The ground speed is $480 \cos \alpha \approx 477.39 \text{ km/h}$.

PROBLEM 10.5 Angle and projection

For $\vec{u} = \langle 3, 4 \rangle$ and $\vec{v} = \langle 5, -1 \rangle$, find the angle between them and the projection of $\vec{u}$ onto $\vec{v}$.

Solution. $\vec{u} \cdot \vec{v} = 15 - 4 = 11$, $\|\vec{u}\| = 5$, $\|\vec{v}\| = \sqrt{26}$. So $\cos \theta = \frac{11}{5\sqrt{26}}$ and $\theta \approx 64.44^\circ$. The projection is $\text{proj}_{\vec{v}} \vec{u} = \frac{11}{26} \langle 5, -1 \rangle = \langle \frac{55}{26}, -\frac{11}{26} \rangle$. The

remainder $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$, which is a quick check.

PROBLEM 10.6 Work done by a slanted force

A sled is pulled 20 m along level ground by a rope at 30° to the horizontal, with tension 50 N. How much work is done?

Solution. Work is $\vec{F} \cdot \vec{d} = \|\vec{F}\| \|\vec{d}\| \cos \theta = 50 \cdot 20 \cos 30^\circ = 500\sqrt{3} \approx 866.03$ J. Only the horizontal component of the pull does work; the vertical component lifts slightly on the sled without moving it along the ground.

PROBLEM 10.7 Distance between polar points

Find the distance between the points with polar coordinates $(3, \frac{\pi}{6})$ and $(5, \frac{2\pi}{3})$.

Solution. The two points and the pole form a triangle with sides 3 and 5 meeting at angle $\frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$. By the Law of Cosines, $d^2 = 9 + 25 - 30 \cos \frac{\pi}{2} = 34$, so $d = \sqrt{34}$. This formula, $d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)$, avoids converting to rectangular form.

PROBLEM 10.8 A hanging weight

A 100 N weight hangs from two cables making angles of 30° and 45° with the horizontal ceiling. Find the tension in each cable.

Solution. At equilibrium the forces sum to zero. Horizontally, $T_1 \cos 30^\circ = T_2 \cos 45^\circ$; vertically, $T_1 \sin 30^\circ + T_2 \sin 45^\circ = 100$. From the first, $T_2 \sin 45^\circ = T_1 \cos 30^\circ$ (because $\sin 45^\circ = \cos 45^\circ$), so $T_1 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 100$ and $T_1 = 100(\sqrt{3} - 1) \approx 73.21$ N. Then $T_2 = \frac{T_1 \cos 30^\circ}{\cos 45^\circ} \approx 89.66$ N. The steeper cable carries more of the load.

SKILL DRILLS

Short problems for building speed and accuracy. Use $r > 0$ and $0 \leq \theta < 2\pi$ for polar coordinates. Answers follow the list.

D10.1 Rectangular coordinates of $(2, \frac{3\pi}{4})$.
 $(4, \frac{\pi}{3})$.

D10.3 Polar coordinates of $(1, 1)$.

D10.2 Rectangular coordinates of

D10.4 Polar coordinates of $(0, -3)$.

- D10.5** Polar coordinates of $(-1, \sqrt{3})$. **D10.16** Eliminate t : $x = \sqrt{t}$, $y = t + 1$.
- D10.6** $r = 5$ in rectangular form. **D10.17** $\| \langle -6, 8 \rangle \|$.
- D10.7** $\theta = \frac{\pi}{4}$ in rectangular form. **D10.18** Unit vector in the direction of $\langle 3, -4 \rangle$.
- D10.8** $r = 6 \cos \theta$ in rectangular form. **D10.19** $2\langle 1, 3 \rangle - \langle 4, -1 \rangle$.
- D10.9** $r \sin \theta = 2$ in rectangular form. **D10.20** Vector of length 10 at direction angle 150° .
- D10.10** $x^2 + y^2 = 4y$ in polar form. **D10.21** $\langle 2, 5 \rangle \cdot \langle 3, -1 \rangle$.
- D10.11** $x = 3$ in polar form. **D10.22** Are $\langle 4, 6 \rangle$ and $\langle -3, 2 \rangle$ orthogonal?
- D10.12** Number of petals of $r = \cos 3\theta$. **D10.23** Angle between $\langle 1, 0 \rangle$ and $\langle 1, 1 \rangle$.
- D10.13** Number of petals of $r = \sin 4\theta$.
- D10.14** Eliminate t : $x = t + 1$, $y = 2t$. **D10.24** $\text{proj}_{\vec{v}} \vec{u}$ for $\vec{u} = \langle 4, 2 \rangle$, $\vec{v} = \langle 1, 0 \rangle$.
- D10.15** Eliminate t : $x = \cos t$, $y = \sin t$.

ANSWERS TO SKILL DRILLS

D10.1 $(2, 2\sqrt{3})$. D10.2 $(-\sqrt{2}, \sqrt{2})$. D10.3 $(\sqrt{2}, \frac{\pi}{4})$. D10.4 $(3, \frac{3\pi}{2})$. D10.5 $(2, \frac{2\pi}{3})$. D10.6 $x^2 + y^2 = 25$. D10.7 $y = x$ (the whole line, using negative r). D10.8 $x^2 + y^2 = 6x$. D10.9 $y = 2$. D10.10 $r = 4 \sin \theta$. D10.11 $r = 3 \sec \theta$. D10.12 3. D10.13 8. D10.14 $y = 2x - 2$. D10.15 $x^2 + y^2 = 1$. D10.16 $y = x^2 + 1$, $x \geq 0$. D10.17 10. D10.18 $\langle \frac{3}{5}, -\frac{4}{5} \rangle$. D10.19 $\langle -2, 7 \rangle$. D10.20 $\langle -5\sqrt{3}, 5 \rangle$. D10.21 1. D10.22 Yes. D10.23 45° . D10.24 $\langle 4, 0 \rangle$.

EXERCISES

COMPUTATION

- 10.1** Convert to rectangular coordinates: $(6, \frac{\pi}{6})$, $(-3, \frac{3\pi}{4})$, $(2, -\frac{\pi}{2})$.
- 10.2** Convert to polar coordinates with $r > 0$, $0 \leq \theta < 2\pi$: $(-3, 3\sqrt{3})$, $(5, -5)$, $(0, -7)$.
- 10.3** Convert to a rectangular equation: $r = 4 \cos \theta$; $r = \sec \theta$; $r^2 \sin 2\theta = 8$.
- 10.4** Convert to a polar equation: $x^2 + y^2 = 9$; $y = x$; $x^2 - y^2 = 1$.
- 10.5** Sketch $r = 2 + 2 \cos \theta$ and $r = 1 - 2 \cos \theta$.
- 10.6** Sketch $r = \cos 2\theta$ and $r = 3 \sin 4\theta$; how many petals does each have?

- 10.7** Find all intersection points of $r = 1 + \cos \theta$ and $r = 3 \cos \theta$.
- 10.8** Eliminate the parameter and sketch: $x = t^2, y = t^3$.
- 10.9** Eliminate the parameter: $x = 1 + \cos t, y = 2 + \sin t$.
- 10.10** Eliminate the parameter: $x = \sec t, y = \tan t$.
- 10.11** Find parametric equations for the line segment from $(-2, 4)$ to $(3, -1)$.
- 10.12** Find parametric equations for the circle with center $(1, -2)$ and radius 3, traced clockwise.
- 10.13** A projectile is fired at 400 ft/s at an angle of 30° . Using $g = 32 \text{ ft/s}^2$, find its range and maximum height.
- 10.14** At what angle should a ball be thrown at 80 ft/s to land 150 ft away (two answers)?
- 10.15** Find $\mathbf{u} + \mathbf{v}$, $3\mathbf{u} - 2\mathbf{v}$, and $|\mathbf{u} - \mathbf{v}|$ for $\mathbf{u} = \langle 2, -5 \rangle$, $\mathbf{v} = \langle -1, 3 \rangle$.
- 10.16** Find the unit vector in the direction of $\langle -6, 8 \rangle$, and the vector of length 10 in the direction of $\langle 1, 2 \rangle$.
- 10.17** Find the magnitude and direction angle of $\mathbf{v} = -4\mathbf{i} + 4\sqrt{3}\mathbf{j}$.
- 10.18** A boat heads across a river at 12 km/h perpendicular to the current, which flows at 5 km/h. Find the boat's actual speed and direction.
- 10.19** A 200-lb crate rests on a ramp inclined at 25° . Find the components of its weight parallel and perpendicular to the ramp.
- 10.20** Find the angle between $\langle 1, 2 \rangle$ and $\langle -3, 4 \rangle$.
- 10.21** Determine whether each pair is parallel, perpendicular, or neither: $\langle 2, -6 \rangle$, $\langle -1, 3 \rangle$; $\langle 4, 1 \rangle$, $\langle -1, 4 \rangle$; $\langle 3, 2 \rangle$, $\langle 2, 3 \rangle$.
- 10.22** Find $\text{proj}_{\mathbf{v}} \mathbf{u}$ for $\mathbf{u} = \langle 4, 2 \rangle$, $\mathbf{v} = \langle 1, -3 \rangle$, and write $\mathbf{u}$ as the sum of a vector parallel and a vector perpendicular to $\mathbf{v}$.
- 10.23** Find the work done by the force $\mathbf{F} = \langle 3, -4 \rangle$ moving an object from $(1, 1)$ to $(6, 9)$.
- 10.24** Find k so that $\langle k, 3 \rangle$ and $\langle 2, -6 \rangle$ are perpendicular.
- 10.25** Two tugboats pull a barge with forces of 2000 lb at 20° and 2500 lb at -15° relative to the barge's axis. Find the resultant.

CONCEPTS AND PROOF

- 10.26** Prove that the polar equation $r = 2a \cos \theta$ is a circle, and find its center and radius.
- 10.27** Show that (r, θ) and $(-r, \theta + \pi)$ represent the same point.
- 10.28** Prove that $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| |\mathbf{v}|$ (the Cauchy–Schwarz inequality).
- 10.29** Prove the triangle inequality $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ using the Cauchy–Schwarz inequality.
- 10.30** Show that the diagonals of a rhombus are perpendicular, using the dot product.
- 10.31** Prove that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is perpendicular to $\mathbf{v}$.
- 10.32** Show that the path $x = a \cos t$, $y = b \sin t$ is an ellipse, and explain how t is related to (but not equal to) the polar angle of the point.
- 10.33** Show that the maximum height of a projectile is $\frac{v_0^2 \sin^2 \alpha}{2g}$.
- 10.34** Explain why the work done by a force perpendicular to the motion is zero, with a physical example.
- 10.35** Find the polar equation of the line through $(2, 0)$ perpendicular to the polar axis, and of the line through $(0, 3)$ parallel to it.

CHALLENGE

- 10.36** Find parametric equations for the cycloid traced by a point on the rim of a wheel of radius r rolling along the x -axis.
- 10.37** Show that the curve $r = \frac{1}{1 - \cos \theta}$ is a parabola, and find its focus and directrix.
- 10.38** A ball thrown with speed v_0 from height h above level ground travels farthest at an angle less than 45° . Find the optimal angle for $v_0 = 20$ m/s and $h = 10$ m numerically.
- 10.39** Find the area of the triangle with vertices $(1, 2)$, $(4, -1)$, $(5, 3)$, using the formula $\frac{1}{2} \sqrt{|\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2}$.
- 10.40** Prove that the medians of a triangle meet at a point two-thirds of the way along each, using vectors.

CHAPTER 11

CONIC SECTIONS

LEARNING OBJECTIVES

- Derive the standard equations of the parabola, ellipse, and hyperbola from their geometric definitions.
- Identify the vertices, foci, axes, eccentricity, and asymptotes of a conic, and graph it.
- Classify and graph shifted conics by completing the square.
- Eliminate the xy -term by rotating the axes, and classify conics with the discriminant.
- Write and interpret conics in polar form using eccentricity.

The conic sections are the curves obtained by slicing a double cone with a plane: circles, ellipses, parabolas, and hyperbolas. The Greeks studied them as pure geometry. Two thousand years later Kepler found that planets move on ellipses, Newton explained why, and engineers learned to focus light, sound, and radio waves with parabolic and elliptical reflectors. Algebraically, the conics are exactly the curves of degree two.

11.1 PARABOLAS

Definition 11.1 (Parabola). A *parabola* is the set of points equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

Theorem 11.2 (Standard parabolas). The parabola with focus $(0, p)$ and directrix $y = -p$ is $x^2 = 4py$. The parabola with focus $(p, 0)$ and directrix $x = -p$ is $y^2 = 4px$. The vertex is the origin.

Proof. A point (x, y) is at distance $\sqrt{x^2 + (y - p)^2}$ from the focus and $|y + p|$ from the directrix. Squaring the equation of the two distances: $x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$, so $x^2 = 4py$. ■

Example 11.3 (Focus and directrix). For $x^2 = -12y$: $4p = -12$, $p = -3$. The focus is $(0, -3)$, the directrix $y = 3$, and the parabola opens downward.

Reflection property: a ray parallel to the axis of a parabola reflects off the curve through the focus. Satellite dishes and telescope mirrors are parabolic for this reason, and a light source at the focus of a headlight produces a parallel beam.

11.2 ELLIPSES

Definition 11.4 (Ellipse). An *ellipse* is the set of points the sum of whose distances from two fixed points (the *foci*) is a constant $2a$.

Theorem 11.5 (Standard ellipse). The ellipse with foci $(\pm c, 0)$ and distance sum $2a$ ($a > c \geq 0$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2 - c^2.$$

Its vertices are $(\pm a, 0)$ (major axis length $2a$), its co-vertices $(0, \pm b)$ (minor axis $2b$), and its eccentricity is $e = \frac{c}{a}$, with $0 \leq e < 1$.

Proof. $\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$. Isolate one radical and square: $(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$, which simplifies to $a\sqrt{(x - c)^2 + y^2} = a^2 - cx$. Squaring again: $a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2$, so $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$. Divide by a^2b^2 . ■

The eccentricity measures how elongated the ellipse is: $e = 0$ is a circle, and e close to 1 is a long thin ellipse. Earth's orbit has $e \approx 0.017$.

Example 11.6 (Reading an ellipse). For $9x^2 + 25y^2 = 225$: divide by 225 to get $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Then $a = 5$, $b = 3$, $c = \sqrt{25 - 9} = 4$: vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$, $e = \frac{4}{5}$.

Reflection property: a ray from one focus reflects off the ellipse through the other focus. Whispering galleries and the lithotripter, which breaks kidney stones with shock waves from one focus aimed at a stone placed at the other, use this.

11.3 HYPERBOLAS

Definition 11.7 (Hyperbola). A *hyperbola* is the set of points the *difference* of whose distances from two foci is a constant $\pm 2a$.

Theorem 11.8 (Standard hyperbola). With foci $(\pm c, 0)$ and difference $2a$ ($c > a > 0$):

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad b^2 = c^2 - a^2.$$

Vertices $(\pm a, 0)$, asymptotes $y = \pm \frac{b}{a}x$, eccentricity $e = \frac{c}{a} > 1$. With the roles of x and y exchanged, the hyperbola opens up and down.

The asymptotes follow from solving for y : $y = \pm \frac{b}{a}x \sqrt{1 - \frac{a^2}{x^2}}$, and the square root approaches 1 as $|x|$ grows. They are the diagonals of the *central box* with corners $(\pm a, \pm b)$, which is the quickest way to sketch a hyperbola.

Example 11.9 (A vertical hyperbola). For $\frac{y^2}{16} - \frac{x^2}{9} = 1$: $a = 4$, $b = 3$, $c = 5$. Vertices $(0, \pm 4)$, foci $(0, \pm 5)$, asymptotes $y = \pm \frac{4}{3}x$.

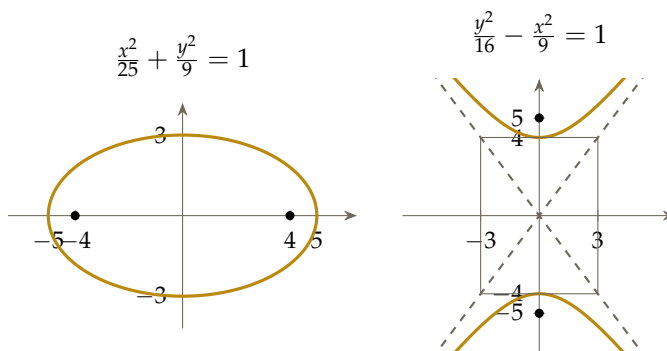


Figure 11.1: An ellipse with its foci, and a hyperbola with its central box, asymptotes, and foci.

11.4 SHIFTED CONICS

Replacing x by $x - h$ and y by $y - k$ shifts a conic so that its center (or vertex) is at (h, k) . Conversely, a second-degree equation without an xy -term can be put in standard form by completing the square.

Example 11.10 (Completing the square). Identify $9x^2 - 4y^2 - 72x - 8y + 176 = 0$.

Solution. $9(x^2 - 8x + 16) - 4(y^2 + 2y + 1) = -176 + 144 - 4$, so $9(x - 4)^2 - 4(y + 1)^2 = -36$, that is

$$\frac{(y + 1)^2}{9} - \frac{(x - 4)^2}{4} = 1.$$

A hyperbola centered at $(4, -1)$, opening up and down, with $a = 3$, $b = 2$, $c = \sqrt{13}$: vertices $(4, 2)$ and $(4, -4)$, foci $(4, -1 \pm \sqrt{13})$, asymptotes $y + 1 = \pm \frac{3}{2}(x - 4)$.

11.5 ROTATION OF AXES

The general second-degree equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \quad (11.1)$$

An xy -term means the conic's axes are tilted. Rotating the coordinate axes through an angle ϕ gives new coordinates related by

$$x = X \cos \phi - Y \sin \phi, \quad y = X \sin \phi + Y \cos \phi.$$

Theorem 11.11 (Eliminating the xy -term). If $B \neq 0$, rotating through the angle ϕ with $\cot 2\phi = \frac{A - C}{B}$ ($0 < \phi < \frac{\pi}{2}$) transforms (11.1) into an equation in X , Y with no XY -term.

Proof. Substituting and collecting, the coefficient of XY is $B(\cos^2 \phi - \sin^2 \phi) + 2(C - A) \sin \phi \cos \phi = B \cos 2\phi - (A - C) \sin 2\phi$. It vanishes exactly when $\cot 2\phi = \frac{A - C}{B}$. ■

Theorem 11.12 (The discriminant). The graph of (11.1), if it is a nondegenerate conic, is a parabola if $B^2 - 4AC = 0$, an ellipse if $B^2 - 4AC < 0$, and a hyperbola

if $B^2 - 4AC > 0$.

The quantity $B^2 - 4AC$ is unchanged by rotation, so it can be computed from the original equation; after a rotation that removes the XY -term it equals $-4A'C'$, whose sign identifies the conic.

Example 11.13 (A rotated hyperbola). Identify $xy = 8$ and find its vertices and foci.

Solution. $A = C = 0, B = 1$: $B^2 - 4AC = 1 > 0$, a hyperbola. $\cot 2\phi = 0$ gives $\phi = \frac{\pi}{4}$. Then $x = \frac{X-Y}{\sqrt{2}}, y = \frac{X+Y}{\sqrt{2}}$, and $xy = \frac{X^2-Y^2}{2} = 8$, so $\frac{X^2}{16} - \frac{Y^2}{16} = 1$: $a = 4, c = 4\sqrt{2}$. The vertices $(X, Y) = (\pm 4, 0)$ correspond to $(x, y) = \pm(2\sqrt{2}, 2\sqrt{2})$, and the foci $(\pm 4\sqrt{2}, 0)$ to $\pm(4, 4)$.

11.6 POLAR EQUATIONS OF CONICS

Theorem 11.14 (Focus–directrix form). Let F be a point, ℓ a line not through F , and $e > 0$. The set of points P with $\frac{|PF|}{\text{dist}(P, \ell)} = e$ is an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. With F at the pole and ℓ the line $x = d$ ($d > 0$), its polar equation is

$$r = \frac{ed}{1 + e \cos \theta}.$$

(Directrix $x = -d$: $r = \frac{ed}{1 - e \cos \theta}$; directrix $y = \pm d$: replace $\cos$ by $\sin$.)

Proof. $|PF| = r$ and the distance from P to $x = d$ is $d - r \cos \theta$. So $r = e(d - r \cos \theta)$, which rearranges to the formula. Converting to rectangular coordinates and completing the square shows the curve is the stated conic. ■

Example 11.15 (A conic in polar form). Identify $r = \frac{2}{1 - \cos \theta}$.

Solution. $e = 1$, a parabola, with $ed = 2$, so $d = 2$ and directrix $x = -2$; the focus is the pole. In rectangular form: $r = 2 + r \cos \theta$, so $\sqrt{x^2 + y^2} = x + 2$, and squaring, $y^2 = 4(x + 1)$: vertex $(-1, 0)$, focus $(0, 0)$, directrix $x = -2$.

LOOKING AHEAD. Newton derived Kepler's first law in polar form: under an inverse-square force, the orbit satisfies $r = \frac{ed}{1+e\cos\theta}$. In Volume III, polar area integrals prove Kepler's second law; in Volume IV, the same inverse-square law in quantum form gives the energy levels of hydrogen. The rotation of axes in Section 11.5 is the first appearance of a deep idea, choosing coordinates in which a quadratic form has no cross terms, that becomes the diagonalization of symmetric matrices.

11.7 WORKED PROBLEMS

Most conic problems come down to two skills: putting an equation into standard form, and reading a , b , c and e from that form.

PROBLEM 11.1 An ellipse from its foci

Find the ellipse with foci $(\pm 3, 0)$ and vertices $(\pm 5, 0)$, and its eccentricity.

Solution. The center is the origin, $a = 5$ and $c = 3$, so $b^2 = a^2 - c^2 = 16$. The equation is $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $e = \frac{c}{a} = \frac{3}{5}$.

PROBLEM 11.2 Completing two squares

Identify $9x^2 + 4y^2 - 36x + 8y + 4 = 0$ and find its center, vertices and foci.

Solution. Group and complete squares: $9(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -4 + 36 + 4$, so $9(x - 2)^2 + 4(y + 1)^2 = 36$ and

$$\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{9} = 1.$$

This is an ellipse centered at $(2, -1)$ with the major axis vertical: $a = 3$, $b = 2$, $c = \sqrt{9 - 4} = \sqrt{5}$. The vertices are $(2, 2)$ and $(2, -4)$; the foci are $(2, -1 \pm \sqrt{5})$. Notice that the right side gains $9 \cdot 4$ and $4 \cdot 1$, not 4 and 1.

PROBLEM 11.3 A hyperbola from its asymptotes

Find the hyperbola with vertices $(0, \pm 4)$ and asymptotes $y = \pm 2x$.

Solution. The vertices are on the y -axis, so the form is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ with $a = 4$, and the asymptotes are $y = \pm \frac{a}{b}x$. So $\frac{4}{b} = 2$ and $b = 2$. The equation is $\frac{y^2}{16} - \frac{x^2}{4} = 1$, with $c = \sqrt{16 + 4} = 2\sqrt{5}$, so the foci are $(0, \pm 2\sqrt{5})$.

PROBLEM 11.4 A satellite dish

A parabolic dish is 2 m across and 0.25 m deep. Where should the receiver be placed?

Solution. Put the vertex at the origin with the dish opening upward: $x^2 = 4py$. The rim passes through $(1, 0.25)$, so $1 = 4p(0.25)$ and $p = 1$. Incoming signals parallel to the axis reflect to the focus, so the receiver goes 1 m above the vertex. The focus can be well outside the dish, as here.

PROBLEM 11.5 Removing the xy -term

Identify $x^2 + xy + y^2 = 3$ and write it in standard form.

Solution. $B^2 - 4AC = 1 - 4 < 0$, so it is an ellipse. The rotation angle satisfies $\cot 2\theta = \frac{A-C}{B} = 0$, so $\theta = 45^\circ$. Substituting $x = \frac{x' - y'}{\sqrt{2}}$, $y = \frac{x' + y'}{\sqrt{2}}$ gives $x^2 + y^2 = x'^2 + y'^2$ and $xy = \frac{x'^2 - y'^2}{2}$, so the equation becomes $\frac{3}{2}x'^2 + \frac{1}{2}y'^2 = 3$, that is,

$$\frac{x'^2}{2} + \frac{y'^2}{6} = 1.$$

The major axis lies along the line $y = -x$ (the y' -axis).

PROBLEM 11.6 A conic in polar form

Identify $r = \frac{6}{2 + \cos \theta}$ and find its vertices, center and directrix.

Solution. Divide by 2: $r = \frac{3}{1 + \frac{1}{2}\cos \theta}$. So $e = \frac{1}{2}$ (an ellipse) and $ed = 3$, giving $d = 6$: the directrix is $x = 6$. The vertices are at $\theta = 0$, $r = 2$, the point $(2, 0)$, and at $\theta = \pi$, $r = 6$, the point $(-6, 0)$. The center is their midpoint $(-2, 0)$, so $a = 4$, $c = 2$ (the focus is the pole), and $b^2 = 16 - 4 = 12$. As a check, $\frac{c}{a} = \frac{1}{2} = e$.

PROBLEM 11.7 Locating a ship

Two radio stations are 100 km apart on an east–west line. A ship receives their signals, and the timing shows it is 60 km closer to station B (east) than to station A (west). The ship is 40 km north of the line through the stations. Where is it?

Solution. Put the stations at $(\pm 50, 0)$. Points whose distances to the two stations differ by 60 form a hyperbola with $2a = 60$ and $c = 50$, so $a = 30$,

$$b^2 = 2500 - 900 = 1600:$$

$$\frac{x^2}{900} - \frac{y^2}{1600} = 1.$$

Being closer to B puts the ship on the right branch. With $y = 40$, $\frac{x^2}{900} = 2$, so $x = 30\sqrt{2} \approx 42.43$. The ship is at about $(42.43, 40)$, measured in km from the midpoint of the stations.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D11.1 Focus of $y^2 = 12x$.

D11.2 Directrix of $x^2 = -8y$.

D11.3 Parabola with vertex at the origin and focus $(0, 5)$.

D11.4 Focus of $(y - 1)^2 = 4(x + 2)$.

D11.5 Vertices of $\frac{x^2}{49} + \frac{y^2}{9} = 1$.

D11.6 Foci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

D11.7 Eccentricity of $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

D11.8 Ellipse with vertices $(0, \pm 6)$ and foci $(0, \pm 4)$.

D11.9 Length of the minor axis of $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

D11.10 Vertices of $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

D11.11 Foci of $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

D11.12 Asymptotes of $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

D11.13 Asymptotes of $\frac{y^2}{4} - \frac{x^2}{25} = 1$.

D11.14 Eccentricity of $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

D11.15 Center of $\frac{(x-3)^2}{4} + \frac{(y+2)^2}{9} = 1$.

D11.16 Classify $x^2 + 4y^2 - 6x = 0$.

D11.17 Classify $x^2 - y^2 + 2x = 3$.

D11.18 Classify $y^2 - 4x + 2y = 0$.

D11.19 Center and radius of $x^2 + y^2 - 4x + 6y = 3$.

D11.20 Classify $xy = 1$.

D11.21 Rotation angle that removes the xy -term from $3x^2 + 2\sqrt{3}xy + y^2 = 4$.

D11.22 Classify $r = \frac{4}{1 - \cos \theta}$.

D11.23 Eccentricity of $r = \frac{10}{2 + 3 \sin \theta}$.

D11.24 Directrix of $r = \frac{4}{1 + \cos \theta}$.

ANSWERS TO SKILL DRILLS

D11.1 $(3, 0)$. D11.2 $y = 2$. D11.3 $x^2 = 20y$. D11.4 $(-1, 1)$. D11.5 $(\pm 7, 0)$.

D11.6 $(\pm 4, 0)$. D11.7 $\frac{4}{5}$. D11.8 $\frac{x^2}{20} + \frac{y^2}{36} = 1$. D11.9 4. D11.10 $(\pm 3, 0)$. D11.11 $(\pm 5, 0)$. D11.12 $y = \pm \frac{4}{3}x$. D11.13 $y = \pm \frac{2}{5}x$. D11.14 $\frac{5}{3}$. D11.15 $(3, -2)$.

D11.16 Ellipse. D11.17 Hyperbola. D11.18 Parabola. D11.19 Center $(2, -3)$, radius 4. D11.20 Hyperbola. D11.21 30° . D11.22 Parabola ($e = 1$). D11.23 $\frac{3}{2}$ (a hyperbola). D11.24 $x = 4$.

EXERCISES

COMPUTATION

- 11.1** Find the focus and directrix of $y^2 = 10x$ and of $x^2 = -6y$.
- 11.2** Find the parabola with vertex at the origin and focus $(0, -\frac{5}{2})$.
- 11.3** Find the parabola with focus $(2, 1)$ and directrix $x = -4$.
- 11.4** A satellite dish is a paraboloid 4 ft across and 1 ft deep. Where should the receiver be placed?
- 11.5** Find the vertices, foci, and eccentricity of $\frac{x^2}{36} + \frac{y^2}{100} = 1$.
- 11.6** Find the vertices, foci, and eccentricity of $4x^2 + y^2 = 16$.
- 11.7** Find the ellipse with foci $(\pm 3, 0)$ and vertices $(\pm 5, 0)$.
- 11.8** Find the ellipse with center at the origin, a vertex at $(0, 7)$, and passing through $(2, 0)$.
- 11.9** Earth's orbit has semimajor axis 1.496×10^8 km and eccentricity 0.0167. Find the perihelion and aphelion distances.
- 11.10** Find the vertices, foci, and asymptotes of $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
- 11.11** Find the vertices, foci, and asymptotes of $9y^2 - x^2 = 36$.
- 11.12** Find the hyperbola with vertices $(\pm 2, 0)$ and asymptotes $y = \pm 3x$.
- 11.13** Find the hyperbola with foci $(0, \pm 8)$ and vertices $(0, \pm 6)$.
- 11.14** Identify and find the key features: $x^2 - 4x + 4y + 16 = 0$.
- 11.15** Identify and find the key features: $4x^2 + 9y^2 - 16x + 54y + 61 = 0$.
- 11.16** Identify and find the key features: $x^2 - y^2 + 6x + 4y + 1 = 0$.
- 11.17** Identify and find the key features: $16x^2 - 9y^2 - 64x - 54y - 161 = 0$.
- 11.18** Use the discriminant to classify: $x^2 + 3xy + y^2 = 4$; $2x^2 - 4xy + 2y^2 = x$; $5x^2 - 6xy + 5y^2 = 8$.

- 11.19** Rotate axes to eliminate the xy -term: $x^2 + 2\sqrt{3}xy - y^2 = 2$. Identify the conic.
- 11.20** Rotate axes to eliminate the xy -term: $5x^2 - 6xy + 5y^2 = 8$. Identify the conic and find its vertices in xy -coordinates.
- 11.21** Identify $r = \frac{6}{2 + \cos \theta}$, and find its eccentricity, directrix, and vertices.
- 11.22** Identify $r = \frac{8}{1 + 3 \sin \theta}$, and find its eccentricity and directrix.
- 11.23** Write the polar equation of the parabola with focus at the pole and directrix $y = 3$.
- 11.24** Write the polar equation of the ellipse with focus at the pole, eccentricity $\frac{1}{2}$, and directrix $x = -4$.
- 11.25** Convert $r = \frac{2}{1 - \cos \theta}$ to rectangular form and identify the vertex, focus, and directrix.

CONCEPTS AND PROOF

- 11.26** Prove that the line through the endpoints of the focal chord perpendicular to the axis of $y^2 = 4px$ (the *latus rectum*) has length $|4p|$.
- 11.27** Prove the formula $b^2 = c^2 - a^2$ for the hyperbola by imitating the proof of Theorem 11.5.
- 11.28** Show that as $e \rightarrow 0$ with a fixed, the ellipse approaches a circle, and as $e \rightarrow 1$ it becomes very elongated.
- 11.29** Show that the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are perpendicular exactly when $a = b$. Such hyperbolas are *rectangular*; $xy = 8$ is one.
- 11.30** Prove that $B^2 - 4AC$ is unchanged by a rotation of axes (verify $B'^2 - 4A'C' = B^2 - 4AC$ from the transformation formulas).
- 11.31** Show that the graph of $x^2 + y^2 + Dx + Ey + F = 0$ is a circle, a point, or empty, and give a condition for each.
- 11.32** Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab , assuming the area of a circle and the effect of scaling on area.
- 11.33** Explain why the polar form $r = \frac{ed}{1 + e \cos \theta}$ makes the focus, not the center, the natural origin for orbits.

- 11.34** Show that the vertices of $r = \frac{ed}{1+e\cos\theta}$ (for $e < 1$) are at $r = \frac{ed}{1+e}$ and $r = \frac{ed}{1-e}$, and deduce that $a = \frac{ed}{1-e^2}$.
- 11.35** Prove the reflection property of the parabola: the tangent line at P makes equal angles with the line from P to the focus and the line through P parallel to the axis.

CHALLENGE

- 11.36** Find the equation of the ellipse with foci $(1, 1)$ and $(5, 1)$ and major axis of length 6.
- 11.37** Find the conic consisting of points whose distance from $(0, 0)$ is twice their distance from the line $x = 3$, and identify it.
- 11.38** Show that the equation $x^2 - 2xy + y^2 - 8x - 8y = 0$ is a parabola, find its axis of symmetry, and find its vertex.
- 11.39** A comet's orbit has perihelion 0.6 AU and eccentricity 0.98. Find its polar equation (focus at the sun) and its aphelion distance.
- 11.40** Find the points where the line $y = x + 1$ meets the ellipse $x^2 + 4y^2 = 16$, and the length of the chord.

CHAPTER 12

SYSTEMS OF EQUATIONS AND MATRICES

LEARNING OBJECTIVES

- Solve systems of linear equations by substitution, elimination, and Gaussian elimination with augmented matrices.
- Recognize inconsistent and dependent systems, and describe infinite solution sets with parameters.
- Add, multiply, and invert matrices, and use inverses to solve systems.
- Compute determinants and apply Cramer's Rule.
- Solve nonlinear systems, and decompose rational expressions into partial fractions.
- Graph systems of inequalities and solve linear programming problems.

Many problems involve several unknowns related by several equations: mixing problems, network flows, balancing chemical equations, fitting a curve through data points. Matrices organize such systems and turn their solution into a mechanical procedure. They are also the language of linear algebra, which in Volume IV becomes the mathematical framework of quantum mechanics.

12.1 LINEAR SYSTEMS

A system of linear equations has either exactly one solution, no solution (it is *inconsistent*), or infinitely many solutions (it is *dependent*). In two variables, each equation is a line, and the three cases are two lines crossing, parallel, or coinciding.

Example 12.1 (Elimination). Solve $3x + 2y = 14$, $x - 2y = 2$.

Solution. Adding the equations eliminates y : $4x = 16$, $x = 4$. Then $4 - 2y = 2$ gives $y = 1$. The solution is $(4, 1)$.

12.1.1 GAUSSIAN ELIMINATION

A system is written compactly as an *augmented matrix*, one row per equation. Three *elementary row operations* produce equivalent systems: (1) add a multiple of one row to another; (2) multiply a row by a nonzero constant; (3) interchange two rows. The goal is *row-echelon form*, in which each row's first nonzero entry (its *leading entry*) is to the right of the one above, and rows of zeros are at the bottom. Then *back-substitution* solves the system from the bottom up.

Example 12.2 (Three equations). Solve $x - y + z = 2$, $2x + y - z = 1$, $x + 2y + 3z = 13$.

Solution.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 13 \end{array} \right] \xrightarrow[\substack{R_2-2R_1 \\ R_3-R_1}]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & -3 \\ 0 & 3 & 2 & 11 \end{array} \right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 5 & 14 \end{array} \right].$$

Back-substitute: $z = \frac{14}{5}$; $3y - 3z = -3$ gives $y = z - 1 = \frac{9}{5}$; and $x = 2 + y - z = 1$. The solution is $(1, \frac{9}{5}, \frac{14}{5})$.

Example 12.3 (A dependent system). Solve $x + 2y - z = 4$, $2x + 5y + z = 9$, $x + 3y + 2z = 5$.

Solution. Row reduction gives $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$. The last row is $0 = 0$:

the third equation adds no information. Let $z = t$ (any real number). Then $y = 1 - 3t$ and $x = 4 - 2y + z = 2 + 7t$. The solutions form a line: $(2 + 7t, 1 - 3t, t)$.

COMMON ERROR (Reading a zero row). A row $[0\ 0\ 0\ |\ 0]$ means the system is dependent (if consistent), while $[0\ 0\ 0\ |\ c]$ with $c \neq 0$ means $0 = c$: no solution. They look alike and mean opposite things.

12.2 MATRIX ALGEBRA

An $m \times n$ matrix has m rows and n columns. Matrices of the same size are added entrywise, and multiplied by scalars entrywise.

Definition 12.4 (Matrix product). If A is $m \times n$ and B is $n \times p$, the product AB is the $m \times p$ matrix whose (i, j) entry is the dot product of row i of A with column j of B : $(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$.

Example 12.5 (Multiplication).

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

Matrix multiplication is *not* commutative: $AB \neq BA$ in general.

Matrix multiplication is associative, $(AB)C = A(BC)$, and distributes over addition. The *identity matrix* I_n has 1s on the diagonal and 0s elsewhere; $AI = IA = A$. A linear system can be written as a single matrix equation $A\mathbf{x} = \mathbf{b}$.

12.3 INVERSES AND DETERMINANTS

Definition 12.6 (Inverse). A square matrix A is *invertible* if there is a matrix A^{-1} with $AA^{-1} = A^{-1}A = I$. Then $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 12.7 (2×2 inverse). $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible exactly when its determinant $ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. Multiply: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I.$ ■

For larger matrices, A^{-1} is found by row-reducing $[A \mid I]$ to $[I \mid A^{-1}]$; if A cannot be reduced to I , it has no inverse.

Definition 12.8 (Determinant). $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. For a 3×3 matrix, expand along the first row:

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Theorem 12.9 (Invertibility). *A square matrix is invertible if and only if its determinant is nonzero. Moreover $\det(AB) = \det A \det B$.*

Geometrically, $|\det A|$ is the factor by which the transformation $\mathbf{x} \mapsto A\mathbf{x}$ scales areas (in two dimensions) or volumes (in three). A determinant of zero means the transformation flattens space, so information is lost and cannot be recovered by an inverse.

Theorem 12.10 (Cramer's Rule). *If $\det A \neq 0$, the solution of $A\mathbf{x} = \mathbf{b}$ is $x_i = \frac{\det A_i}{\det A}$, where A_i is A with column i replaced by $\mathbf{b}$.*

Example 12.11 (Cramer's Rule). Solve $2x - 3y = 7$, $4x + 5y = 3$.

Solution. $\det A = 10 + 12 = 22$. $x = \frac{1}{22} \begin{vmatrix} 7 & -3 \\ 3 & 5 \end{vmatrix} = \frac{44}{22} = 2$, $y = \frac{1}{22} \begin{vmatrix} 2 & 7 \\ 4 & 3 \end{vmatrix} = \frac{-22}{22} = -1$.

12.4 NONLINEAR SYSTEMS AND PARTIAL FRACTIONS

Example 12.12 (A line and a circle). Solve $x^2 + y^2 = 25$, $x + y = 7$.

Solution. Substitute $y = 7 - x$: $x^2 + (7 - x)^2 = 25$, $2x^2 - 14x + 24 = 0$, $x^2 - 7x + 12 = 0$, so $x = 3$ or 4 . The solutions are $(3, 4)$ and $(4, 3)$.

12.4.1 PARTIAL FRACTIONS

Adding fractions is easy; calculus often needs the reverse, writing a rational function as a sum of simpler ones. The setup depends on how the denominator factors: each linear factor $(ax + b)^k$ contributes terms $\frac{A_1}{ax+b} + \cdots + \frac{A_k}{(ax+b)^k}$, and each irreducible quadratic factor $(ax^2 + bx + c)^k$ contributes $\frac{B_1x+C_1}{ax^2+bx+c} + \cdots$. The unknown constants solve a linear system.

Example 12.13 (Distinct linear factors). Decompose $\frac{5x+7}{x^2+2x-3}$.

Solution. $x^2 + 2x - 3 = (x+3)(x-1)$. Write $\frac{5x+7}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$, so $5x+7 = A(x-1) + B(x+3)$. Setting $x = 1$: $12 = 4B$, $B = 3$. Setting $x = -3$: $-8 = -4A$, $A = 2$. So the expression equals $\frac{2}{x+3} + \frac{3}{x-1}$.

Example 12.14 (A repeated factor). Decompose $\frac{x^2+1}{x(x-1)^2}$.

Solution. $\frac{x^2+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, so $x^2+1 = A(x-1)^2 + Bx(x-1) + Cx$. $x = 0$: $A = 1$. $x = 1$: $C = 2$. Comparing x^2 coefficients: $1 = A + B$, so $B = 0$. Result: $\frac{1}{x} + \frac{2}{(x-1)^2}$.

12.5 SYSTEMS OF INEQUALITIES AND LINEAR PROGRAMMING

12.5.1 GRAPHING SYSTEMS OF INEQUALITIES

The graph of a linear inequality such as $2x + y \leq 8$ is a *half-plane*: draw the boundary line $2x + y = 8$ (solid for $\leq$ or $\geq$, dashed for $<$ or $>$), then test one point not on the line, usually the origin. Here $2(0) + 0 \leq 8$ is true, so the half-plane containing the origin is shaded. The solution set of a *system* of inequalities is the intersection of the half-planes, a region whose corners are found by solving pairs of boundary equations.

Example 12.15 (A region and its corners). Graph the system $x \geq 0$, $y \geq 0$, $x + y \leq 12$, $x + 2y \leq 16$, and find its corners.

Solution. Each inequality keeps the side of its line containing the origin, so the region is the quadrilateral in Fig. 12.1. Its corners are $(0, 0)$; $(12, 0)$, where $x + y = 12$ meets the x -axis; $(0, 8)$, where $x + 2y = 16$ meets the

y -axis; and the intersection of the two slanted lines. Subtracting $x + y = 12$ from $x + 2y = 16$ gives $y = 4$, so $x = 8$: the corner $(8, 4)$. The point $(12, 0)$ satisfies $x + 2y \leq 16$, and $(0, 8)$ satisfies $x + y \leq 12$, so both are genuine corners.

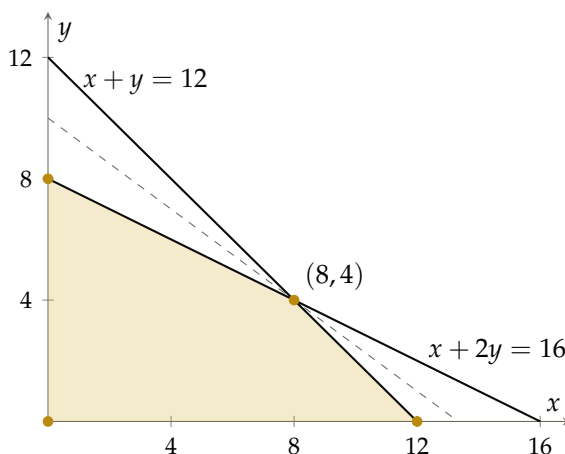


Figure 12.1: The feasible region of Example 12.15. The dashed line is the level line $30x + 40y = 400$, which touches the region only at the corner $(8, 4)$.

12.5.2 LINEAR PROGRAMMING

A *linear programming* problem asks for the largest or smallest value of a linear *objective function*, such as $P = 30x + 40y$, over a region defined by linear inequalities, the *feasible region*. The lines $30x + 40y = P$ are parallel; as P increases they slide across the plane, and the last point of the region they touch is a corner. That is the reason for the following theorem.

Theorem 12.16 (Corner Point Theorem). *If a linear objective function has a maximum or minimum value on a feasible region bounded by lines, that value occurs at a corner of the region. If the region is bounded, both the maximum and the minimum exist.*

So the method is: graph the region, find the corners, evaluate the objective at each corner, and compare.

Example 12.17 (Maximizing profit). A workshop builds desks and bookshelves. A desk takes 1 hour of cutting and 1 hour of finishing; a bookshelf

takes 1 hour of cutting and 2 hours of finishing. Each week there are 12 hours of cutting and 16 hours of finishing available. A desk earns \$30 profit and a bookshelf \$40. How many of each should be built?

Solution. With x desks and y bookshelves, the constraints are exactly those of Example 12.15, and the objective is $P = 30x + 40y$. At the corners:

$$P(0,0) = 0, \quad P(12,0) = 360, \quad P(8,4) = 400, \quad P(0,8) = 320.$$

The maximum weekly profit is \$400, from 8 desks and 4 bookshelves. Notice that the best plan uses both resources fully: $8 + 4 = 12$ and $8 + 2(4) = 16$.

Example 12.18 (Minimizing cost on an unbounded region). Minimize $C = 3x + 2y$ subject to $x \geq 0$, $y \geq 0$, $x + y \geq 6$, $x + 3y \geq 10$.

Solution. Here the shaded sides are away from the origin, so the region is unbounded, with corners $(0,6)$, $(10,0)$ and the intersection of $x + y = 6$ with $x + 3y = 10$, which is $(4,2)$. Then $C(0,6) = 12$, $C(4,2) = 16$, $C(10,0) = 30$. Because both coefficients of C are positive, C only grows as one moves out into the region, so the minimum exists: $C = 12$ at $(0,6)$. On the same region, C has no maximum.

COMMON ERROR. Evaluate the objective only at corners of the *feasible* region. An intersection of two boundary lines that violates a third constraint is not a corner, and its value is meaningless. Always check each candidate against every inequality.

12.5.3 APPLICATIONS OF LINEAR SYSTEMS

Linear systems also determine curves from data. A parabola $y = ax^2 + bx + c$ has three unknown coefficients, so three points determine it; a circle $x^2 + y^2 + Dx + Ey + F = 0$ also has three unknowns.

Example 12.19 (A parabola through three points). Find the parabola through $(1,4)$, $(2,9)$ and $(-1,6)$.

Solution. Substituting each point into $y = ax^2 + bx + c$ gives

$$a + b + c = 4, \quad 4a + 2b + c = 9, \quad a - b + c = 6.$$

Subtracting the third from the first gives $2b = -2$, so $b = -1$. Then $a +$

$c = 5$ and $4a + c = 11$, so $3a = 6$, $a = 2$ and $c = 3$. The parabola is $y = 2x^2 - x + 3$; check $(2, 9)$: $8 - 2 + 3 = 9$.

Example 12.20 (A circle through three points). Find the circle through $(0, 0)$, $(6, 0)$ and $(0, 8)$.

Solution. In $x^2 + y^2 + Dx + Ey + F = 0$, the point $(0, 0)$ gives $F = 0$; $(6, 0)$ gives $36 + 6D = 0$, so $D = -6$; $(0, 8)$ gives $64 + 8E = 0$, so $E = -8$. Completing squares in $x^2 - 6x + y^2 - 8y = 0$ gives $(x - 3)^2 + (y - 4)^2 = 25$: center $(3, 4)$, radius 5. The center is the midpoint of the hypotenuse from $(6, 0)$ to $(0, 8)$, as it must be for a right triangle.

Example 12.21 (A mixture). A chemist needs 10 L of a 32% acid solution and has 20% and 50% solutions on hand. How much of each should be mixed?

Solution. Let x and y be the liters of 20% and 50% solution. Volume gives $x + y = 10$; acid gives $0.20x + 0.50y = 0.32(10) = 3.2$. Substituting $x = 10 - y$: $2 + 0.3y = 3.2$, so $y = 4$ and $x = 6$. Use 6 L of the 20% solution and 4 L of the 50% solution.

LOOKING AHEAD. In Volume III, partial fractions are the method for integrating every rational function. In Volume IV, matrices take center stage: a quantum observable is a Hermitian matrix, its possible measured values are the eigenvalues λ satisfying $\det(A - \lambda I) = 0$, and Gaussian elimination is how the corresponding eigenvectors are found.

12.6 WORKED PROBLEMS

These problems cover elimination, matrices and nonlinear systems. Check every solution by substituting into all of the original equations, not just some of them.

PROBLEM 12.1 Elimination with three unknowns

Solve $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$.

Solution. Row-reduce the augmented matrix. $R_2 - 2R_1$ and $R_3 - R_1$ give

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -2 & -4 \end{array} \right].$$

Swap R_2 and R_3 , then $R_3 + 3R_2$: the last row becomes $0 \ 0 \ -7 \mid -21$, so $z = 3$. Back-substitute: $y - 2(3) = -4$, $y = 2$; $x + 2 + 3 = 6$, $x = 1$. The solution is $(1, 2, 3)$, and all three equations check.

PROBLEM 12.2 Infinitely many solutions

Solve $x + y + z = 4$, $x - y + 2z = 1$, $2x + 3z = 5$.

Solution. The third equation is the sum of the first two, so it adds no information, and the system has infinitely many solutions. Subtracting the second equation from the first gives $2y - z = 3$. Let $z = t$: then $y = \frac{3+t}{2}$ and $x = 4 - y - z = \frac{5-3t}{2}$. The solution set is $\left\{ \left(\frac{5-3t}{2}, \frac{3+t}{2}, t \right) \right\}$, a line in space. For example, $t = 1$ gives $(1, 2, 1)$.

PROBLEM 12.3 Solving with an inverse matrix

Solve $A\vec{x} = \vec{b}$ for $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 8 \\ 15 \end{bmatrix}$.

Solution. Row-reducing $[A \mid I]$ to $[I \mid A^{-1}]$ gives

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}, \quad \vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -44 + 16 + 30 \\ -16 + 0 + 15 \\ 24 - 8 - 15 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Check $AA^{-1} = I$ for at least one row and column. Here $\det A = 1$, which is why the inverse has integer entries.

PROBLEM 12.4 A determinant by cofactors

Evaluate $\begin{vmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix}$.

Solution. Expand along the first row: $2 \begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 2(-8) - 1(-4) + 3(1) = -9$. Expanding along a row or column with a zero saves work: down the first column the answer is $2(-8) + 1(4 + 3) = -9$ as well.

PROBLEM 12.5 A symmetric nonlinear system

Solve $x^2 + y^2 = 10$, $xy = 3$.

Solution. Combine the equations: $(x + y)^2 = x^2 + y^2 + 2xy = 16$ and $(x - y)^2 = 10 - 6 = 4$. So $x + y = \pm 4$ and $x - y = \pm 2$, and the four sign choices give $(3, 1)$, $(1, 3)$, $(-1, -3)$, $(-3, -1)$. Geometrically, a circle meets a hyperbola in four points.

PROBLEM 12.6 Partial fractions with a quadratic factor

Decompose $\frac{3x^2 + x + 2}{(x + 1)(x^2 + 3)}$.

Solution. The quadratic $x^2 + 3$ is irreducible, so write $\frac{A}{x+1} + \frac{Bx+C}{x^2+3}$, giving $3x^2 + x + 2 = A(x^2 + 3) + (Bx + C)(x + 1)$. At $x = -1$: $4 = 4A$, so $A = 1$. Comparing x^2 coefficients: $3 = A + B$, so $B = 2$. Comparing constants: $2 = 3A + C$, so $C = -1$. The result is $\frac{1}{x+1} + \frac{2x-1}{x^2+3}$.

PROBLEM 12.7 Ticket sales

A theater sold 500 tickets for \$4400. Adult tickets cost \$12, child tickets \$6 and senior tickets \$8, and twice as many adult tickets as senior tickets were sold. How many of each were sold?

Solution. With a, c, s the numbers sold: $a + c + s = 500$, $12a + 6c + 8s = 4400$, $a = 2s$. Substituting $a = 2s$: $c + 3s = 500$ and $6c + 32s = 4400$. Then $c = 500 - 3s$, so $3000 - 18s + 32s = 4400$, $s = 100$. So 200 adult, 200 child and 100 senior tickets.

PROBLEM 12.8 When is the solution unique?

For which k does $kx + y = 1$, $x + ky = 1$ have exactly one solution, infinitely many, or none?

Solution. The coefficient determinant is $k^2 - 1$, so there is exactly one solution when $k \neq \pm 1$ (it is $x = y = \frac{1}{k+1}$). If $k = 1$, both equations are $x + y = 1$: infinitely many solutions. If $k = -1$, the equations are $-x + y = 1$ and $x - y = 1$, which add to $0 = 2$: no solution.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

D12.1 Solve $x + y = 7, x - y = 1$.

D12.2 Solve $2x + 3y = 12, x = 3$.

D12.3 Solve $3x - y = 5, x + 2y = 4$.

D12.4 How many solutions:
 $x + 2y = 3, 2x + 4y = 6$?

D12.5 How many solutions:
 $x + 2y = 3, 2x + 4y = 5$?

D12.6 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}$.

D12.7 $3 \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$.

D12.8 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

D12.9 $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

D12.10 Size of AB if A is 2×3 and B is 3×4 .

D12.11 $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.

D12.12 $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

D12.13 Inverse of $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

D12.14 Inverse of $\begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

D12.15 For which k is $\begin{bmatrix} k & 2 \\ 3 & 6 \end{bmatrix}$ not invertible?

D12.16 Solve $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

D12.17 Solve $y = x^2, y = x + 2$.

D12.18 Solve $x^2 + y^2 = 25, y = 3$.

D12.19 Solve $xy = 6, x + y = 5$.

D12.20 Decompose $\frac{2}{x(x+2)}$.

D12.21 Decompose $\frac{x+5}{(x-1)(x+2)}$.

D12.22 Set up (do not solve) the decomposition of $\frac{1}{x^2(x+1)}$.

D12.23 Corners of $x \geq 0, y \geq 0, 2x + y \leq 6$.

D12.24 Maximum of $P = x + 3y$ on the region in D12.23.

ANSWERS TO SKILL DRILLS

D12.1 $(4, 3)$. D12.2 $(3, 2)$. D12.3 $(2, 1)$. D12.4 Infinitely many. D12.5 None.
 D12.6 $\begin{bmatrix} 1 & 1 \\ 5 & 9 \end{bmatrix}$. D12.7 $\begin{bmatrix} 6 & -3 \\ 0 & 12 \end{bmatrix}$. D12.8 $\begin{bmatrix} 17 \\ 39 \end{bmatrix}$. D12.9 $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$. D12.10 2×4 .
 D12.11 2. D12.12 24. D12.13 $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$. D12.14 $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 2 \end{bmatrix}$. D12.15 $k = 1$.
 D12.16 $(1, -1)$. D12.17 $(-1, 1), (2, 4)$. D12.18 $(\pm 4, 3)$. D12.19 $(2, 3), (3, 2)$.
 D12.20 $\frac{1}{x} - \frac{1}{x+2}$. D12.21 $\frac{2}{x-1} - \frac{1}{x+2}$. D12.22 $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. D12.23 $(0, 0), (3, 0), (0, 6)$. D12.24 18, at $(0, 6)$.

EXERCISES

COMPUTATION

- 12.1** Solve $2x + 5y = 1, 3x - 2y = 11$.
- 12.2** Solve $\frac{x}{2} + \frac{y}{3} = 2, \frac{2x}{5} - y = -\frac{11}{5}$ by first clearing fractions.
- 12.3** Determine whether the system $4x - 6y = 10, 6x - 9y = 15$ is inconsistent or dependent, and describe its solutions.
- 12.4** A chemist mixes a 10% acid solution with a 30% solution to make 200 mL of 25% solution. How much of each is used?
- 12.5** Solve by Gaussian elimination: $x + y + z = 6, 2x - y + z = 3, x + 2y - z = 2$.
- 12.6** Solve by Gaussian elimination: $x - 2y + 3z = 1, 2x - 3y + 5z = 4, x + y = 11$.
- 12.7** Solve: $x + y - z = 2, 2x + 3y - z = 5, 3x + 4y - 2z = 7$.
- 12.8** Solve: $x + 2y + z = 3, 2x + 5y + 3z = 7, x + 3y + 3z = 5$.
- 12.9** Find the parabola $y = ax^2 + bx + c$ through $(1, 2), (2, 7), (-1, 4)$.
- 12.10** Find $A + B, 2A - 3B, AB$, and BA for $A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.
- 12.11** Compute $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & -1 \end{bmatrix}$.
- 12.12** Find the inverse of $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and use it to solve $5x + 3y = 4, 3x + 2y = 1$.

12.13 Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by row reduction.

12.14 Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 5 & 2 & -2 \end{vmatrix}$.

12.15 For which k is $\begin{bmatrix} k & 2 \\ 8 & k \end{bmatrix}$ not invertible?

12.16 Use Cramer's Rule to solve $3x + 4y = 5$, $2x - y = 7$.

12.17 Use Cramer's Rule to find z for $x + y + z = 2$, $2x - y + 3z = 9$, $x + 2y - z = -3$.

12.18 Find the area of the triangle with vertices $(1, 2)$, $(4, 6)$, $(7, 1)$ using $\frac{1}{2} |\det|$ of the matrix with rows $(x_i, y_i, 1)$.

12.19 Solve $y = x^2 - 4$, $y = 2x - 1$.

12.20 Solve $x^2 + y^2 = 10$, $xy = 3$.

12.21 Solve $x^2 - y^2 = 5$, $x^2 + 2y^2 = 8$.

12.22 Decompose $\frac{x + 14}{x^2 - x - 6}$.

12.23 Decompose $\frac{3x^2 - x + 4}{x^3 + 2x}$.

12.24 Decompose $\frac{2x + 3}{(x + 1)^2}$.

12.25 Decompose $\frac{x^3 + x^2 + 2}{x^2 - 1}$ (divide first).

CONCEPTS AND PROOF

12.26 Prove that a system of two linear equations in two unknowns has exactly one solution if and only if the determinant of its coefficient matrix is nonzero.

12.27 Give an example of 2×2 matrices A, B with $AB = 0$ but $A \neq 0$ and $B \neq 0$. Why can't this happen with real numbers?

12.28 Show that $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices.

- 12.29** Show that $\det \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = 1$, and interpret: this matrix rotates the plane, preserving area.
- 12.30** Prove that swapping two rows of a 2×2 matrix changes the sign of its determinant.
- 12.31** Show that a homogeneous system $A\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$, and that it has others exactly when $\det A = 0$.
- 12.32** Explain why Gaussian elimination never changes the solution set of a system.
- 12.33** Show that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of the line through (x_1, y_1) and (x_2, y_2) .
- 12.34** Find all 2×2 matrices that commute with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- 12.35** If $A^2 = I$, what are the possible values of $\det A$?

CHALLENGE

- 12.36** Find the cubic $y = ax^3 + bx^2 + cx + d$ through $(0, 1)$, $(1, 0)$, $(-1, 4)$, $(2, 13)$.
- 12.37** Find A^n for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$; how are the entries of the second related to the Fibonacci numbers?
- 12.38** Balance the chemical equation $a \text{C}_3\text{H}_8 + b \text{O}_2 \rightarrow c \text{CO}_2 + d \text{H}_2\text{O}$ by setting up and solving a linear system.
- 12.39** Evaluate the determinant of the $n \times n$ matrix with 2 on the diagonal, -1 just above and below it, and 0 elsewhere, for $n = 2, 3, 4$, and guess the pattern.
- 12.40** Find all real solutions of $x^2 + y^2 = 13$, $x^2 - xy + y^2 = 7$.

CHAPTER 13

SEQUENCES, SERIES, AND INDUCTION

LEARNING OBJECTIVES

- Work with sequences defined explicitly and recursively, and with sigma notation.
- Find terms and sums of arithmetic and geometric sequences, and sum infinite geometric series.
- Prove statements about integers by mathematical induction.
- Expand powers of binomials with the Binomial Theorem and Pascal's triangle.

A sequence is a function whose inputs are the positive integers: a list $a_1, a_2, a_3, \dots$ that goes on forever. Sequences describe anything that happens in steps: monthly loan balances, generations of a population, the iterations of an algorithm. Adding the terms of a sequence produces a series. This chapter covers the two most important kinds, arithmetic and geometric, introduces the idea of an infinite sum, and develops the method of proof by induction, which is how statements about all positive integers are established.

13.1 SEQUENCES AND SUMMATION NOTATION

A sequence can be given by a formula for the n th term, such as $a_n = \frac{n}{n+1}$ (giving $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$), or *recursively*, by giving the first term(s) and a rule for each term in terms of earlier ones. The *Fibonacci sequence* is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$: 1, 1, 2, 3, 5, 8, 13, ...

Definition 13.1 (Sigma notation). $\sum_{k=1}^n a_k = a_1 + a_2 + \cdots + a_n$. The letter k is the *index of summation*; any letter may be used.

Sigma notation obeys the rules $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k$ and $\sum ca_k = c \sum a_k$, and $\sum_{k=1}^n c = nc$.

Example 13.2 (Partial sums and telescoping). $\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}$. Everything except the first and last terms cancels.

13.2 ARITHMETIC SEQUENCES

Definition 13.3 (Arithmetic sequence). An *arithmetic sequence* has a constant difference d between consecutive terms: $a_n = a_1 + (n-1)d$.

Theorem 13.4 (Arithmetic sum). $S_n = \sum_{k=1}^n a_k = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1)d]$.

Proof. Write the sum forward and backward and add: each of the n pairs $a_1 + a_n$, $a_2 + a_{n-1}$, $\dots$ has the same total $a_1 + a_n$. So $2S_n = n(a_1 + a_n)$. ■

The story goes that the young Gauss summed $1 + 2 + \cdots + 100$ this way in seconds: 50 pairs of 101, total 5050.

Example 13.5 (Seats in a theater). A theater has 20 seats in the first row, and each row has 3 more than the one before. How many seats are in 30 rows?

Solution. $a_1 = 20$, $d = 3$, $a_{30} = 20 + 29 \cdot 3 = 107$. $S_{30} = \frac{30}{2}(20 + 107) = 1905$.

13.3 GEOMETRIC SEQUENCES AND SERIES

Definition 13.6 (Geometric sequence). A geometric sequence has a constant ratio r between consecutive terms: $a_n = a_1 r^{n-1}$.

Theorem 13.7 (Geometric sum). For $r \neq 1$, $S_n = \sum_{k=1}^n a_1 r^{k-1} = a_1 \frac{1 - r^n}{1 - r}$.

Proof. $S_n - rS_n = (a_1 + a_1 r + \cdots + a_1 r^{n-1}) - (a_1 r + \cdots + a_1 r^n) = a_1 - a_1 r^n$; divide by $1 - r$. ■

Example 13.8 (An annuity). \$200 is deposited at the end of each month for 10 years into an account earning 6% per year compounded monthly. Find the final balance.

Solution. The last deposit earns no interest, the one before earns one month's, and so on; the first earns 119 months'. With $i = 0.005$, the balance is $200[1 + 1.005 + \cdots + 1.005^{119}] = 200 \cdot \frac{1.005^{120} - 1}{0.005} \approx \$32,775.87$.

13.3.1 INFINITE GEOMETRIC SERIES

What should $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ mean? Its partial sums are $S_n = 2(1 - \frac{1}{2^n})$, which approach 2 as n grows. We define the sum of an infinite series to be the limit of its partial sums, when that limit exists.

Theorem 13.9 (Infinite geometric series). If $|r| < 1$, then $\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r}$.
If $|r| \geq 1$ (and $a_1 \neq 0$), the series has no sum.

Proof. In $S_n = a_1 \frac{1 - r^n}{1 - r}$, the term r^n approaches 0 when $|r| < 1$. When $|r| \geq 1$, the terms $a_1 r^{k-1}$ do not approach 0, so the partial sums cannot settle. ■

Example 13.10 (Repeating decimals). $0.\overline{36} = \frac{36}{100} + \frac{36}{100^2} + \cdots = \frac{36/100}{1 - 1/100} = \frac{36}{99} = \frac{4}{11}$.

Example 13.11 (A bouncing ball). A ball dropped from 10 m rebounds to $\frac{3}{4}$ of its previous height each time. Find the total distance it travels.

Solution. Down 10, then up and down 7.5, then up and down 5.625, ...:
 $D = 10 + 2(7.5 + 5.625 + \cdots) = 10 + 2 \cdot \frac{7.5}{1 - \frac{3}{4}} = 10 + 60 = 70$ m.

COMMON ERROR (Applying the formula when $|r| \geq 1$). Plugging $r = 2$ into $\frac{a_1}{1-r}$ “shows” that $1 + 2 + 4 + 8 + \cdots = -1$. The formula is valid only for $|r| < 1$; for $|r| \geq 1$ the partial sums grow without bound or oscillate.

13.4 MATHEMATICAL INDUCTION

Theorem 13.12 (Principle of Mathematical Induction). Let $P(n)$ be a statement about the positive integer n . If (1) $P(1)$ is true, and (2) for every $k \geq 1$, $P(k)$ implies $P(k+1)$, then $P(n)$ is true for all positive integers n .

Think of an infinite row of dominoes: (1) knocks over the first, and (2) guarantees each falling domino knocks over the next. In step (2), the assumption that $P(k)$ is true is the *induction hypothesis*.

Example 13.13 (A sum formula). Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$.

Solution. *Base case.* For $n = 1$: $1 = \frac{1 \cdot 2 \cdot 3}{6}$. *Inductive step.* Assume the formula for $n = k$. Then

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6}. \end{aligned}$$

Since $2k^2 + 7k + 6 = (k+2)(2k+3)$, this is $\frac{(k+1)(k+2)(2(k+1)+1)}{6}$, the formula for $n = k+1$.

Example 13.14 (Divisibility). Prove that $4^n - 1$ is divisible by 3 for all $n \geq 1$.

Solution. $4^1 - 1 = 3$. If $4^k - 1 = 3m$, then $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(3m + 1) - 1 = 12m + 3 = 3(4m + 1)$.

COMMON ERROR (Skipping the base case). The inductive step alone proves nothing. “If $k = k + 1$ then $k + 1 = k + 2$ ” is a valid implication, yet the statement $n = n + 1$ is false for every n ; it has no base case.

13.5 THE BINOMIAL THEOREM

Definition 13.15 (Binomial coefficients). For integers $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n! = 1 \cdot 2 \cdots n$ and $0! = 1$.

Theorem 13.16 (Binomial Theorem). For every positive integer n ,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n.$$

Proof. Expanding $(a + b)(a + b) \cdots (a + b)$, each term chooses a or b from each of the n factors. The term $a^{n-k} b^k$ arises once for each way of choosing which k factors contribute b , and there are $\binom{n}{k}$ such ways. ■

The coefficients satisfy Pascal’s rule $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, which generates Pascal’s triangle row by row:

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & 1 & & 1 & \\ & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \\ & 1 & & 4 & & 6 & & 4 & & 1 \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Example 13.17 (Expansions). $(2x - y)^4 = 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$. The term containing x^5 in $(x^2 + \frac{1}{x})^{10}$: the general term is $\binom{10}{k} (x^2)^{10-k} x^{-k} = \binom{10}{k} x^{20-3k}$, and $20 - 3k = 5$ gives $k = 5$: the term is $252x^5$.

LOOKING AHEAD. Infinite series are a central topic of Volume III: they represent functions such as $e^x = \sum \frac{x^n}{n!}$ and $\sin x = x - \frac{x^3}{3!} + \cdots$, and the ge-

ometric series is the prototype for every convergence test. The binomial theorem extends to non-integer exponents as an infinite series, $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$, which gives the low-speed approximation of Einstein's kinetic energy formula. Induction is used throughout Volumes III and IV to prove statements for all n .

13.6 WORKED PROBLEMS

These problems use the sequence and series formulas, induction and the Binomial Theorem. In each, first decide what kind of sequence or sum is involved.

PROBLEM 13.1 An arithmetic sequence from two terms

An arithmetic sequence has $a_5 = 17$ and $a_{12} = 45$. Find a_1 , d and the sum of the first 20 terms.

Solution. Seven steps of d take a_5 to a_{12} , so $7d = 28$ and $d = 4$. Then $a_1 = a_5 - 4d = 1$. The sum is $S_{20} = \frac{20}{2}(2a_1 + 19d) = 10(2 + 76) = 780$.

PROBLEM 13.2 A geometric sequence from two terms

A geometric sequence has $a_3 = 12$ and $a_6 = 96$. Find a_1 , r and S_{10} .

Solution. Dividing, $r^3 = \frac{96}{12} = 8$, so $r = 2$, and $a_1 = \frac{a_3}{r^2} = 3$. Then $S_{10} = a_1 \frac{r^{10}-1}{r-1} = 3(1024 - 1) = 3069$.

PROBLEM 13.3 A repeating decimal

Write $0.\overline{27} = 0.272727 \dots$ as a fraction.

Solution. $0.\overline{27} = \frac{27}{100} + \frac{27}{100^2} + \cdots$, a geometric series with first term $\frac{27}{100}$ and ratio $\frac{1}{100}$. Its sum is $\frac{\frac{27}{100}}{1-\frac{1}{100}} = \frac{27}{99} = \frac{3}{11}$. Dividing 3 by 11 confirms it.

PROBLEM 13.4 A bouncing ball

A ball is dropped from 10 m and each bounce reaches $\frac{3}{4}$ of the previous height. What total vertical distance does it travel?

Solution. It falls 10 m, then rises and falls 7.5 m, then 5.625 m twice, and so on. The total is

$$10 + 2(7.5 + 5.625 + \cdots) = 10 + 2 \cdot \frac{7.5}{1 - \frac{3}{4}} = 10 + 60 = 70 \text{ m.}$$

Infinitely many bounces cover a finite distance, and (by a similar series for the times) take a finite time.

PROBLEM 13.5 Induction with divisibility

Prove that $5^n - 1$ is divisible by 4 for every positive integer n .

Solution. *Base case.* $5^1 - 1 = 4$, which is divisible by 4.

Inductive step. Assume $5^k - 1 = 4m$ for some integer m . Then

$$5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(5^k - 1) + 4 = 5(4m) + 4 = 4(5m + 1),$$

which is divisible by 4. By induction, the statement holds for all $n \geq 1$. The key move is rewriting the $(k + 1)$ case so the k case appears inside it.

PROBLEM 13.6 One binomial coefficient

Find the coefficient of x^5 in $(2x - 3)^8$.

Solution. The general term is $\binom{8}{k}(2x)^k(-3)^{8-k}$. For x^5 , take $k = 5$: $\binom{8}{5}2^5(-3)^3 = 56 \cdot 32 \cdot (-27) = -48384$. Track the sign carefully; it comes from the odd power of -3 .

PROBLEM 13.7 The constant term

Find the term of $\left(x^2 + \frac{1}{x}\right)^9$ that does not contain x .

Solution. The general term is $\binom{9}{k}(x^2)^{9-k}\left(\frac{1}{x}\right)^k = \binom{9}{k}x^{18-3k}$. The exponent is 0 when $k = 6$, so the constant term is $\binom{9}{6} = 84$.

PROBLEM 13.8 Saving every month

You deposit \$200 at the end of each month into an account paying 6% annual interest compounded monthly. How much is there after 10 years?

Solution. The monthly rate is 0.005 and there are 120 deposits. The first deposit grows for 119 months, the last for none, so the balance is the geometric sum

$$200 \left(1 + 1.005 + 1.005^2 + \cdots + 1.005^{119} \right) = 200 \cdot \frac{1.005^{120} - 1}{0.005} \\ \approx \$32,775.87.$$

Of this, \$24,000 is deposits and about \$8,776 is interest.

SKILL DRILLS

Short problems for building speed and accuracy. Answers follow the list.

- D13.1** a_{10} if $a_n = 3n - 2$. **D13.13** $5 - \frac{5}{3} + \frac{5}{9} - \cdots$.
- D13.2** First four terms of $a_n = \frac{(-1)^n}{n}$. **D13.14** Does $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k$ converge?
- D13.3** $\sum_{k=1}^5 k^2$. **D13.15** $0.\overline{3}$ as a fraction.
- D13.4** $\sum_{k=1}^4 (2k + 1)$. **D13.16** $0.\overline{45}$ as a fraction.
- D13.5** General term of 4, 7, 10, ... **D13.17** 6!.
- D13.6** Twentieth term of 2, 5, 8, ... **D13.18** $\binom{7}{3}$.
- D13.7** $1 + 2 + \cdots + 100$. **D13.19** $\sum_{k=0}^5 \binom{5}{k}$.
- D13.8** Sum of the first 15 terms of 3, 7, 11, ... **D13.20** Expand $(x + 1)^3$.
- D13.9** Common ratio of 6, $-3, \frac{3}{2}, \dots$ **D13.21** Expand $(a - b)^4$.
- D13.10** Eighth term of 1, 2, 4, ... **D13.22** Fourth term of $(x + 2)^6$ (in descending powers of x).
- D13.11** $1 + 3 + 9 + \cdots + 3^6$. **D13.23** Coefficient of x^2 in $(1 + x)^{10}$.
- D13.12** $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$. **D13.24** a_4 if $a_1 = 2$ and $a_{n+1} = 3a_n - 1$.

ANSWERS TO SKILL DRILLS

D13.1 28. D13.2 $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}$. D13.3 55. D13.4 24. D13.5 $a_n = 3n + 1$.
D13.6 59. D13.7 5050. D13.8 465. D13.9 $-\frac{1}{2}$. D13.10 128. D13.11 1093.
D13.12 2. D13.13 $\frac{15}{4}$. D13.14 No ($|r| > 1$). D13.15 $\frac{1}{3}$. D13.16 $\frac{5}{11}$. D13.17 720.
D13.18 35. D13.19 32. D13.20 $x^3 + 3x^2 + 3x + 1$. D13.21

$$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4. \quad \text{D13.22 } 160x^3. \quad \text{D13.23 } 45. \quad \text{D13.24 } 41.$$

EXERCISES

COMPUTATION

- 13.1** Write the first five terms of $a_n = \frac{(-1)^n}{2^n}$ and of $a_1 = 3, a_{n+1} = 2a_n - 1$.
- 13.2** Find a formula for the n th term of $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots$ and of $2, 5, 10, 17, 26, \dots$
- 13.3** Evaluate $\sum_{k=1}^5 k^2$ and $\sum_{j=0}^4 (-2)^j$.
- 13.4** Find $\sum_{k=1}^{99} (\sqrt{k+1} - \sqrt{k})$.
- 13.5** Find the 40th term and the sum of the first 40 terms of the arithmetic sequence $7, 11, 15, \dots$
- 13.6** The 5th term of an arithmetic sequence is 18 and the 12th is 46. Find the first term and the 30th term.
- 13.7** How many terms of $3, 7, 11, \dots$ must be added to get 820?
- 13.8** Find the sum of all three-digit multiples of 7.
- 13.9** Find the 8th term of the geometric sequence $5, -10, 20, \dots$, and the sum of its first 8 terms.
- 13.10** The 2nd term of a geometric sequence is 6 and the 5th is 162. Find the first term and the ratio.
- 13.11** Find $\sum_{k=1}^{10} 3 \left(\frac{1}{2}\right)^{k-1}$.
- 13.12** Find the sum: $8 - 4 + 2 - 1 + \dots$.
- 13.13** Find the sum: $\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n}$.
- 13.14** Express $2.1\overline{45}$ as a fraction.
- 13.15** For which x does $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ converge, and what is its sum?
- 13.16** A \$15,000 car loan at 6% annual interest, compounded monthly, is repaid over 5 years. Find the monthly payment R using $15000 = R \frac{1 - (1.005)^{-60}}{0.005}$.
- 13.17** A pendulum swings 30 cm on its first swing and each swing is 90% of the previous one. Find the total distance traveled before it stops.

13.18 Expand $(x + 2)^5$.

13.19 Expand $(3a - b)^4$.

13.20 Find the coefficient of x^7 in $(2x + 1)^{10}$.

13.21 Find the constant term in $(x - \frac{2}{x})^8$.

13.22 Use the binomial theorem to approximate $(1.01)^{10}$ to five decimal places.

CONCEPTS AND PROOF

13.23 Prove by induction: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

13.24 Prove by induction: $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

13.25 Prove by induction: $\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}$.

13.26 Prove by induction that $n^3 - n$ is divisible by 6 for every positive integer n .

13.27 Prove by induction that $2^n > n^2$ for all $n \geq 5$.

13.28 Prove that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for the Fibonacci numbers.

13.29 Prove Pascal's rule $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ algebraically.

13.30 Prove that $\sum_{k=0}^n \binom{n}{k} = 2^n$ and $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$.

13.31 Explain why an infinite geometric series with $|r| \geq 1$ cannot have a sum, using the terms themselves.

13.32 Find the error in this "proof" that all horses are the same color: "A single horse is one color. If any k horses are one color, take $k + 1$ horses; the first k are one color, and so are the last k , and they overlap, so all $k + 1$ are one color."

CHALLENGE

13.33 Find $\sum_{k=1}^n k \cdot 2^k$ in closed form, and prove your formula by induction.

13.34 Prove Bernoulli's inequality: $(1 + x)^n \geq 1 + nx$ for every $x \geq -1$ and every positive integer n .

13.35 Prove that $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ (Binet's formula).

- 13.36** A square of side 1 has a second square inscribed by joining the midpoints of its sides, then a third inside the second, forever. Find the total area of all squares.
- 13.37** Show that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing for $n = 1, \dots, 6$ numerically, and use the Binomial Theorem to show $a_n < 3$ for all n .

APPENDIX A

FORMULA REFERENCE

This appendix collects the algebra, geometry, and trigonometry used most often in this book, for quick reference.

A.1 ALGEBRA

A.1.1 EXPONENTS AND RADICALS

For $a, b > 0$ and real r, s :

$$a^r a^s = a^{r+s}, \quad \frac{a^r}{a^s} = a^{r-s}, \quad (a^r)^s = a^{rs},$$

$$(ab)^r = a^r b^r, \quad a^{-r} = \frac{1}{a^r}, \quad a^{m/n} = \sqrt[n]{a^m}.$$

A.1.2 FACTORING

$$a^2 - b^2 = (a - b)(a + b), \quad a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2),$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}).$$

A.1.3 THE BINOMIAL THEOREM

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

A.1.4 QUADRATIC FORMULA

The roots of $ax^2 + bx + c = 0$ ($a \neq 0$) are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. They are real and distinct if $b^2 - 4ac > 0$, a double root if $b^2 - 4ac = 0$, and complex conjugates if $b^2 - 4ac < 0$.

A.1.5 INEQUALITIES AND ABSOLUTE VALUE

$|a| = \sqrt{a^2}$; $|ab| = |a||b|$; $|x| < c \iff -c < x < c$; and the *triangle inequality* $|a + b| \leq |a| + |b|$, with its consequence $||a| - |b|| \leq |a - b|$.

A.1.6 LOGARITHMS

For $a > 0, a \neq 1$: $\log_a x = y \iff a^y = x$. For $x, y > 0$:

$$\log_a(xy) = \log_a x + \log_a y, \quad \log_a \frac{x}{y} = \log_a x - \log_a y,$$

$$\log_a(x^r) = r \log_a x, \quad \log_a x = \frac{\ln x}{\ln a}.$$

A.2 GEOMETRY

Circle	$A = \pi r^2, \quad C = 2\pi r$
Sector	$A = \frac{1}{2}r^2\theta, \quad \text{arc } s = r\theta$
Triangle	$A = \frac{1}{2}bh = \frac{1}{2}ab \sin C$
Sphere	$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$
Cylinder	$V = \pi r^2 h, \quad \text{lateral } S = 2\pi r h$
Cone	$V = \frac{1}{3}\pi r^2 h, \quad \text{lateral } S = \pi r \sqrt{r^2 + h^2}$

Distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Circle with center (h, k) and radius r : $(x - h)^2 + (y - k)^2 = r^2$. Lines: $y - y_1 = m(x - x_1)$; perpendicular lines have slopes with product -1 .

A.3 TRIGONOMETRY

A.3.1 DEFINITIONS AND EXACT VALUES

For a point (x, y) at distance r from the origin on the terminal side of θ : $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, $\tan \theta = \frac{y}{x}$, $\csc \theta = \frac{r}{y}$, $\sec \theta = \frac{r}{x}$, $\cot \theta = \frac{x}{y}$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

A.3.2 IDENTITIES

Pythagorean: $\sin^2 \theta + \cos^2 \theta = 1$, $1 + \tan^2 \theta = \sec^2 \theta$, $1 + \cot^2 \theta = \csc^2 \theta$.

Symmetry: $\sin(-\theta) = -\sin \theta$, $\cos(-\theta) = \cos \theta$, $\sin(\frac{\pi}{2} - \theta) = \cos \theta$, $\sin(\pi - \theta) = \sin \theta$.

Addition:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

Double angle: $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$.

Half angle (power reduction): $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$.

Product to sum:

$$\begin{aligned}\sin A \cos B &= \frac{1}{2}[\sin(A - B) + \sin(A + B)], \\ \sin A \sin B &= \frac{1}{2}[\cos(A - B) - \cos(A + B)], \\ \cos A \cos B &= \frac{1}{2}[\cos(A - B) + \cos(A + B)].\end{aligned}$$

Laws of sines and cosines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, $c^2 = a^2 + b^2 - 2ab \cos C$.

A.3.3 INVERSE TRIGONOMETRIC FUNCTIONS

Function	Domain	Range
$\arcsin x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

APPENDIX B

THE UNIT CIRCLE

The table lists the exact values of sine, cosine, and tangent at the sixteen standard angles in $[0, 2\pi)$. The point on the unit circle at angle θ is $(\cos \theta, \sin \theta)$. Each value was checked by computer algebra.

Degrees	Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	—
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
180°	π	0	-1	0
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270°	$\frac{3\pi}{2}$	-1	0	—
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$

SIGNS BY QUADRANT

	I	II	III	IV
sin, csc	+	+	−	−
cos, sec	+	−	−	+
tan, cot	+	−	+	−

ANSWERS TO SELECTED EXERCISES

Answers are given for odd-numbered exercises that have a short final answer. Every answer here was checked by computer algebra. Proofs are not answered; check a proof against the definitions and theorems it uses.

CHAPTER 1

NUMBERS, EQUATIONS, AND INEQUALITIES

- 1.1** $[-2, 5]; (-\infty, -1] \cup (3, \infty); (2, 4)$
1.3 $x = 6$
1.5 $x = 3 \pm \sqrt{7}$
1.7 $k = \pm 6$
1.9 No solution ($x = 3$ is extraneous)
1.11 $x = 2$ ($x = -3$ is extraneous)
1.13 $x = 3$ or $x = -\frac{7}{3}$; no solution
1.15 $(-\infty, -3]$
1.17 $[1, 6]$
1.19 $(-3, 4)$
1.21 $[-2, 0] \cup [2, \infty)$
1.23 $(-4, 3)$
1.25 10; midpoint $(1, 1)$
1.27 Center $(4, -1)$, radius 3
1.29 $(x - 2)^2 + (y + 1)^2 = 25$
1.37 $c > 4$
1.41 $x = -3$ or $x = 2$
1.43 Every x in $[5, 10]$
1.45 Nine ordered pairs: $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, $(12, 12)$, and the reversals of the first four.

CHAPTER 2

FUNCTIONS AND THEIR GRAPHS

- 2.1** $15; 2a^2 + a; 8a^2 - 6a + 1; 4a^2 - 6a + 2$
2.3 $[-2, 3]$
2.5 $6x + 3h - 1$
2.7 $\frac{1}{\sqrt{x+h} + \sqrt{x}}$
2.9 Odd; even; neither; odd; neither
2.11 $-\frac{1}{ab}$
2.13 A V with vertex $(2, -1)$; an inverted V with vertex $(-1, 1)$
2.15 $(3, 17); (-1, 5)$
2.17 $f \circ g = 2x^2 - 1; g \circ f = 4x^2 - 12x + 10; f \circ f = 4x - 9$
2.19 $F = f \circ g$ with $g(x) = \sqrt{x}$, $f(x) = \sqrt{1+x}$; $G = f \circ g$ with $g(x) = x^2 + 1$, $f(x) = x^{-3}$
2.21 $f^{-1}(x) = \frac{x+7}{4}$
2.23 $f^{-1}(x) = \frac{2x}{1-x}$; domain $x \neq 1$, range $y \neq -2$
2.25 $f^{-1}(3) = 1; f(f^{-1}(2)) = 2$
2.37 $f(f(x)) = \frac{x-1}{x}$, $f(f(f(x))) = x$; since $2026 = 3 \cdot 675 + 1$, the answer is $f(x)$.
2.39 $f(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}$

CHAPTER 3

POLYNOMIAL AND RATIONAL FUNCTIONS

- 3.1** $f(x) = -3(x-2)^2 + 5$; vertex $(2, 5)$;
range $(-\infty, 5]$
- 3.3** 42 ft at $t = 1.5$ s
- 3.5** \$15
- 3.7** 0 (mult. 2, touches), 3 (mult. 1), -1
(mult. 3, crosses and flattens)
- 3.9** Quotient $x^2 - 2x - 1$, remainder $5x - 4$
- 3.11** -713
- 3.13** $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$
- 3.15** $(x-3)(x-2)(2x+1)$
- 3.17** $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i; \pm 2, \pm 2i$
- 3.19** $x^4 - 6x^3 + 11x^2 - 6x + 10$
- 3.21** Hole at $(2, \frac{3}{4})$; vertical asymptote $x = -2$; horizontal $y = 1$; intercepts $(-1, 0)$
and $(0, \frac{1}{2})$
- 3.23** Vertical $x = 0$; slant $y = x - 1$; hole
at $x = -1$
- 3.25** $r(x) = \frac{3(x-1)^2}{x^2-4}$ (one possible answer)
- 3.33** Slant asymptote $y = x + 1$; it is
crossed at $(-1, 0)$
- 3.37** $a^2 - 2b$
- 3.39** For $n > 1$ both factors $n^2 \pm 2n + 2$ exceeded 1.

CHAPTER 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

- 4.1** Asymptotes $y = 0, y = -3, y = 0$
- 4.3** \$2860.90; \$2866.66
- 4.5** 4, $-3, -3, \frac{1}{2}, 7, 9$
- 4.7** $2 \log x + \log(x-1) - \frac{1}{2} \log(x+1)$
- 4.9** $\ln \frac{x^3 \sqrt{x-1}}{(x^2+1)^2}$
- 4.11** 2.2619; -3.3219

$$\mathbf{4.13} \quad x = \frac{2 \ln 7}{\ln 7 - \ln 4} \approx 6.954$$

$$\mathbf{4.15} \quad x = \ln \frac{2}{3} \approx -0.405$$

$$\mathbf{4.17} \quad x = 20 \quad (x = -5 \text{ is extraneous})$$

$$\mathbf{4.19} \quad x = e^4 \text{ or } x = e^{-1}$$

$$\mathbf{4.21} \quad 15.75 \text{ mg; about } 99.7 \text{ years}$$

$$\mathbf{4.23} \quad \text{About } 22.4 \text{ minutes}$$

$$\mathbf{4.25} \quad 100 \text{ times; about } 73.0 \text{ dB}$$

$$\mathbf{4.41} \quad x = 64$$

$$\mathbf{4.43} \quad x = 8, y = 4$$

$$\mathbf{4.45} \quad A = 9, k = \ln \frac{3}{2} \approx 0.405; P = 500 \text{ at } t = \frac{\ln 9}{\ln 1.5} \approx 5.42$$

CHAPTER 5

ANGLES AND THE TRIGONOMETRIC FUNCTIONS

$$\mathbf{5.1} \quad \frac{2\pi}{5}, -\frac{5\pi}{6}, \frac{13\pi}{4}; 105^\circ, -300^\circ, \approx 114.59^\circ$$

$$\mathbf{5.3} \quad 6\pi \approx 18.85 \text{ cm}$$

$$\mathbf{5.5} \quad \text{About } 1036.7 \text{ mi}$$

$$\mathbf{5.7} \quad \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right); \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right); \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$\mathbf{5.9} \quad \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{3}, -2$$

$$\mathbf{5.11} \quad -\frac{\sqrt{2}}{2}, \frac{1}{2}, -\sqrt{3}$$

$$\mathbf{5.13} \quad \cos \theta = -\frac{15}{17}, \csc \theta = \frac{17}{8}$$

$$\mathbf{5.15} \quad \tan t = \frac{\sin t}{\sqrt{1 - \sin^2 t}}, \text{ since } \cos t > 0 \text{ in quadrant IV}$$

$$\mathbf{5.17} \quad \text{Legs } \approx 11.47 \text{ and } \approx 16.38; \text{ other angle } 55^\circ$$

$$\mathbf{5.19} \quad \text{About } 28.53 \text{ ft}$$

$$\mathbf{5.21} \quad \text{About } 114.9 \text{ m}$$

$$\mathbf{5.23} \quad \frac{3\sqrt{3}}{2}r^2$$

$$\mathbf{5.25} \quad 5\pi \approx 15.7 \text{ cm; } 15\pi \approx 47.1 \text{ cm}^2$$

$$\mathbf{5.37} \quad \sin 18^\circ = \frac{\sqrt{5}-1}{4}$$

$$\mathbf{5.39} \quad \text{About } 58.14 \text{ cm}$$

CHAPTER 6

TRIGONOMETRIC GRAPHS AND INVERSE FUNC-

TIONS

6.1 Amplitude 4, period $\frac{2\pi}{3}$, phase shift $-\frac{\pi}{3}$, midline $y = 2$

6.3 $y = 5 \sin 3 \left(x + \frac{\pi}{4} \right)$

6.5 $2; \frac{\pi}{4}; 6\pi$

6.9 $y = 6 \cos(4\pi t)$

6.11 $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$

6.13 $\frac{2}{5}, \frac{5\pi}{6}, -\frac{\pi}{4}$

6.15 $\frac{25}{24}, \frac{3\sqrt{2}}{4}$

6.17 $\frac{1-x^2}{1+x^2}$

6.19 $x = \frac{5}{2}$

6.31 24π

6.33 $x = \frac{\sqrt{17}-3}{4}$

CHAPTER 7

TRIGONOMETRIC IDENTITIES AND EQUATIONS

7.7 1

7.9 $\sqrt{2} \sin \left(x - \frac{\pi}{4} \right); 13 \sin(x + \phi)$ with $\phi = \arctan \frac{12}{5}$

7.11 $\frac{2\sqrt{2}}{3}, -\frac{1}{3}, -2\sqrt{2}$

7.13 $\frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$

7.15 $2 \sin 5x \cos 2x; -2 \sin 3x \sin 2x$

7.17 $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

7.19 $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

7.21 $\frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

7.23 $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

7.25 $0, \frac{2\pi}{3}$

7.41 $-\frac{1}{8}$

7.43 $\frac{89}{2}$

CHAPTER 8

SOLVING TRIANGLES

8.1 $C = 62^\circ, a \approx 199.55, b \approx 241.52$

8.3 None: $h = 12 \sin 30^\circ = 6 > 5$

8.5 Two triangles: $B \approx 61.85^\circ, C \approx 82.15^\circ,$

$c \approx 13.48$; or $B \approx 118.15^\circ, C \approx 25.85^\circ,$
 $c \approx 5.93$

8.7 None: $h = 25 \sin 40^\circ \approx 16.07 > 10$

8.9 $A \approx 34.05^\circ, B \approx 44.42^\circ, C \approx 101.54^\circ$

8.11 ≈ 84.57

8.13 Diagonals ≈ 10.58 and ≈ 17.44 ; area
 $48\sqrt{3} \approx 83.14$

8.15 ≈ 2.00 km

8.17 Area ≈ 8966.6 m²; largest angle $\approx 94.94^\circ$

8.19 $\approx 68.69^\circ$

8.41 $r = 4$

8.43 The sides are 3, 4, 5: area 6; angles
 $90^\circ, \approx 36.87^\circ, \approx 53.13^\circ$

CHAPTER 9

COMPLEX NUMBERS

9.1 $3 - 10i; 11 - 10i; 2 + 2i$

9.3 $i; i; 1$

9.5 $5; 1; 25$

9.7 The circle of radius 3 centered at $2i$; the
imaginary axis

9.9 $-3 + 3\sqrt{3}i; 1 - i$

9.11 16

9.13 $-\frac{1}{64}$

9.15 $\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -3i$

9.17 $1 + \sqrt{3}i, -\sqrt{3} + i, -1 - \sqrt{3}i, \sqrt{3} - i$

9.19 $2^{1/6} (\cos \phi + i \sin \phi)$ with $\phi =$
 $\frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}$

9.41 $(1 + i)^{100} = -2^{50}$, so the alternat-
ing sum of even binomial coefficients is
 -2^{50} .

9.43 With $u = \omega + \omega^{-1} = 2 \cos \frac{2\pi}{5}$, the
equation becomes $u^2 + u - 1 = 0$, so
 $u = \frac{\sqrt{5}-1}{2}$.

9.45 It is the real part of $\sum \omega^k = 0$.

CHAPTER 10

POLAR COORDINATES, PARAMETRIC EQUATIONS, AND VECTORS

10.1 $(3\sqrt{3}, 3); \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right); (0, -2)$

10.3 $x^2 + y^2 = 4x; x = 1; xy = 4$

10.5 A cardioid; a limaçon with an inner loop

10.7 $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, -\frac{\pi}{3}\right)$, and the pole

10.9 $(x-1)^2 + (y-2)^2 = 1$

10.11 $x = -2 + 5t, y = 4 - 5t, 0 \leq t \leq 1$

10.13 Range ≈ 4330.1 ft; maximum height 625 ft

10.15 $\langle 1, -2 \rangle; \langle 8, -21 \rangle; \sqrt{73}$

10.17 Magnitude 8; direction 120°

10.19 ≈ 84.52 lb parallel; ≈ 181.26 lb perpendicular

10.21 Parallel; perpendicular; neither

10.23 -17

10.25 ≈ 4294.4 lb at $\approx 0.49^\circ$ to the axis

10.43 About 39.3°

CHAPTER 11

CONIC SECTIONS

11.1 Focus $\left(\frac{5}{2}, 0\right)$, directrix $x = -\frac{5}{2}$; focus $\left(0, -\frac{3}{2}\right)$, directrix $y = \frac{3}{2}$

11.3 $(y-1)^2 = 12(x+1)$

11.5 Vertices $(0, \pm 10)$, foci $(0, \pm 8)$, $e = \frac{4}{5}$

11.7 $\frac{x^2}{25} + \frac{y^2}{16} = 1$

11.9 Perihelion $\approx 1.471 \times 10^8$ km; aphelion $\approx 1.521 \times 10^8$ km

11.11 $\frac{y^2}{4} - \frac{x^2}{36} = 1$: vertices $(0, \pm 2)$, foci $(0, \pm 2\sqrt{10})$, asymptotes $y = \pm \frac{x}{3}$

11.13 $\frac{y^2}{36} - \frac{x^2}{28} = 1$

11.15 Ellipse $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$: center $(2, -3)$, vertices $(-1, -3)$ and $(5, -3)$, foci $(2 \pm \sqrt{5}, -3)$

11.17 Hyperbola $\frac{(x-2)^2}{9} - \frac{(y+3)^2}{16} = 1$: center $(2, -3)$, vertices $(-1, -3)$, $(5, -3)$, foci $(-3, -3)$, $(7, -3)$

11.19 With $\phi = 30^\circ$: $X^2 - Y^2 = 1$, a hyperbola

11.21 Ellipse, $e = \frac{1}{2}$, directrix $x = 6$; vertices $(2, 0)$ and $(-6, 0)$

11.23 $r = \frac{3}{1 + \sin \theta}$

11.25 $y^2 = 4(x+1)$: vertex $(-1, 0)$, focus $(0, 0)$, directrix $x = -2$

11.41 $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{5} = 1$

11.43 After rotating by 45° : $Y^2 = 4\sqrt{2}X$. The vertex is the origin and the axis is the line $y = x$.

11.45 $\left(\frac{-4 \pm 2\sqrt{19}}{5}, \frac{1 \pm 2\sqrt{19}}{5}\right)$; chord length $\frac{4\sqrt{38}}{5}$

CHAPTER 12

SYSTEMS OF EQUATIONS AND MATRICES

12.1 $(3, -1)$

12.3 Dependent (the second equation is $\frac{3}{2}$ times the first): all points $(t, \frac{2t-5}{3})$

12.5 $(1, 2, 3)$

12.7 Dependent: $(1+2t, 1-t, t)$

12.9 $y = 2x^2 - x + 1$

12.11 $\begin{bmatrix} 8 & -1 \\ 1 & 1 \end{bmatrix}$

12.13 $\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

12.15 $k = \pm 4$

12.17 $z = 2$ (the full solution is $(1, -1, 2)$)

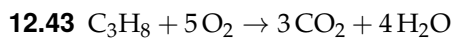
12.19 $(-1, -3)$ and $(3, 5)$

12.21 $(\pm\sqrt{6}, \pm 1)$, all four sign choices

12.23 $\frac{2}{x} + \frac{x-1}{x^2+2}$

12.25 $x+1 + \frac{2}{x-1} - \frac{1}{x+1}$

12.41 $y = 2x^3 + x^2 - 4x + 1$



12.45 $(2, 3), (3, 2), (-2, -3), (-3, -2)$

CHAPTER 13

SEQUENCES, SERIES, AND INDUCTION

13.1 $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}; 3, 5, 9, 17, 33$

13.3 55; 11

13.5 $a_{40} = 163; S_{40} = 3400$

13.7 20 terms

13.9 $a_8 = -640; S_8 = -425$

13.11 $\frac{3069}{512}$

13.13 $\frac{4}{3}$

13.15 For $|x| < 3$; the sum is $\frac{3}{3-x}$

13.17 300 cm

13.19 $81a^4 - 108a^3b + 54a^2b^2 - 12ab^3 + b^4$

13.21 1120

13.33 $(n-1)2^{n+1} + 2$

13.37 Each term $\binom{n}{k} \frac{1}{n^k} \leq \frac{1}{k!} \leq \frac{1}{2^{k-1}}$, so

$$a_n < 1 + \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 3.$$